Blue Pelican

Calculus Enrichment Topics

Student Version 1.01

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Enrichment Topic A

(Special sine and cosine limits)

Demonstrate that the following limit is true:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

Begin with the infinite series expansion definition for $\sin(x_{rad})$:

$$\sin(x_{rad}) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ ...$$

For very small x, **sin(x) ≈ x**.

Demonstrate that the following limit is true:

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0$$

Begin with the infinite series expansion definition for $\cos(x_{rad})$:

$$\cos(x_{rad}) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ ...$$

For very small x, **cos(x) ≈ 1– x^2/2!**

Enrichment Topic B

(Formal definition of continuity)

Continuity at a point x = a: A function *f(x)* is continuous at *x = a* if:

(1) $f(a)$ exists,

(2) $\lim_{x \to a} f(x)$ exists, and

(3) $\lim_{x \to a} f(x) = f(a)$

Example 1: Draw an example of a function in which (1) is violated *at x = a* but is otherwise continuous.

Example 2: Draw an example of a function in which (2) is violated at *x = a* but is otherwise continuous.

Example 3: Draw an example of a function in which (3) is violated at *x = a* but is otherwise continuous.

Enrichment Topic C

Verification of the power rule

Power rule:

$f(x) = x^n$

$f'(x) = nx^{n-1}$;where *n* can be a positive integer, a negative integer, or fractional

To derive this rule we use the fundamental definition of the derivative and expand $(x + \Delta x)^n$ with a binomial expansion:

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Enrichment Topic D

(Derivation of product & quotient rules)

Product rule:

If f(x) = u(x) v(x), then

f'(x) = u · v' + v · u'

To derive this rule we use the fundamental definition of the derivative and add 0 in the form of $-u(x + \Delta x)\, v(x) + u(x + \Delta x)\, v(x)$:

Quotient rule:

$$\text{If } f(x) = \frac{u}{v} \text{ then} \qquad \mathbf{f'(x) = \frac{v \cdot u' - u \cdot v'}{v^2}}$$

To derive this rule we use the fundamental definition of the derivative, clear the complex fraction, and then add 0 in the form of
–u(x)·v(x) + u(x)·v(x):

Enrichment Topic E

(Derivative of sine and cosine)

To derive f(x) = sin(x) we use the following four items:

- $f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$
- sin(A + B) = sin(A) cos(B) + sin(B) cos(A)
- $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$
- $\lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta} = 0$

(The last two items are from **Enrichment topic A**.)

To derive f(x) = cos(x) we use the following four items:

- $$f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
- cos(A + B) = cos(A) cos(B) – sin(A) sin(B)
- $$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$
- $$\lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta} = 0$$

  (The last two items are from **Enrichment topic A**.)

Enrichment Topic F

(Verification of the Chain Rule)

The Chain Rule:

If y = f(g(x)) and both f(x) and g(x) are differentiable, then:

y' = f '(g(x)) g'(x)

The astute observer may have noticed a flaw in the above "proof". A change in *x* (called Δx) induces a change in *g* (called Δg) which in turn causes a change in *y* (called Δy). Notice that Δg appears in the denominator and if it is zero, the proof fails.

The proof is not rigorous since we must consider the possibility of *g* not changing (Δg = 0) as *x* changes. Fortunately the rule prevails under more rigorous scrutiny. However, for most beginning students the above "proof" is sufficient.

Enrichment Topic G

Verification of exponential derivatives

Show that $\frac{d}{dx}a^u = a^u \ln(a)\frac{du}{dx}$:

Begin with $y = a^u$ (*u* is a function of *x*)

Enrichment Topic H

Verification of logarithm derivatives

Show that $\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e$

Begin with $\frac{d}{dx}\log_a x = \lim_{\Delta x \to 0} \frac{\log_a(x + \Delta x) - \log_a(x)}{\Delta x}$

Enrichment Topic I

Verification of inverse trig function derivatives

Show that $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$

Begin with $y = \text{Sin}^{-1}(x)$ $\quad -\pi/2 \le y \le \pi/2$

Show that $\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$

Begin with $y = \text{Cos}^{-1}(x)$ $\quad 0 \le y \le \pi$

Show that $\frac{d}{dx}\arctan(x) = \frac{1}{1 + x^2}$

Begin with $y = \text{Tan}^{-1}(x)$ $\quad -\pi/2 \le y \le \pi/2$

Enrichment Topic J

An argument in support of the Fundamental Theorem of Calculus

Consider the following functions from physics:

position s(t), velocity v(t), and acceleration a(t)

The derivative of s(t) produces v(t) and the derivative of v(t) produces a(t).

Similarly, integrating a(t) produces v(t) and integrating v(t) produces s(t).

Now consider this specific displacement function s(t) and its derivative over the time interval [2 sec, 5 sec]:

$s(t) = (3t^2 + 4)$ meters
$s'(t) = v(t) = 6t$ meters/sec

The **net displacement** in this interval is simply the difference in displacement at t = 2 sec and t = 5 sec.

disp = $s(5) – s(2) = 3(5)^2 + 4 – (3(2)^2 + 4) = 63$ meters

Notice that the **area** under the velocity curve given by $\int_2^5 6t\,dt$ is

Finally, notice these two answers are the same. Therefore, the meaning of this definite integral is the accumulated distance traveled during the time interval:

$$\int_2^5 v(t)dt = s(5) - s(2)$$

Generalizing from this we have the **Fundamental Theorem of Calculus**:

If f(x) is continuous over [a,b], differentiable in (a,b), and F'(x) = f(x) then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Example: Evaluate $\int_{-1}^{4} 2t^2 dt$.

Enrichment Topic K

Why the absolute value for the integral of 1/x?

When integrating $\int \frac{du}{u}$ it is customary to write the solution as ln|u| +C.

Why the absolute value? Superficially, it seems to be an easy answer, "Because it's illegal to take the logarithm of a negative number."

Probing a little deeper, let's take a closer look at the reverse of this process… taking the derivative of f(x) = ln(x).

First, assuming that x > 0, find the derivative of f(x) = ln(x):

which implies:

Now assume that x < 0 and find the derivative of f(x) = ln(–x) (Notice we must use –x here so that we are taking the log of a positive number.):

which implies:

Surprisingly, the two derivatives above are the same. So what are we to do when we must integrate $\int \frac{1}{x}$ dx and *x* could be any value, positive or negative? **Both** answer above (ln(x) when x > 0 and ln(–x) when x < 0) can **equivalently** written as ln|x|.

So in general we write:

$$\int \frac{du}{u} = \ln|u| + c$$

Enrichment Topic L

(Partial Fractions)

The technique of "partial fractions" is the **process of decomposing a given rational expression into a sum of fractions with denominators of lower degree.** For example,

$$\frac{x-11}{2x^2+5x-3}=\frac{2}{x+3}-\frac{3}{2x-1}$$

The virtue in doing this is that the two fractions on the right are **easier to handle** than is the larger and more complex one on the left. As a specific example, integrating the fraction on the left is difficult whereas the two fractions on the right are each easy to integrate. . . they both produce natural logs.

There are four different cases to consider depending on the **degree** of factors of the denominator and their **repetition**:

Case 1: When the denominator can be resolved into real linear factors, all of which are distinct. . . see Example 1.

Case 2: When the denominator can be resolved into real linear factors, some of which are repeated. . . see Example 2.

Case 3: When the denominator contains at least one quadratic factor but no repeated quadratic factor. . . see Example 3.

Case 4. When the denominator contains at least one repeated quadratic factor. . . see Example 4.

Example 1: (case 1) Resolve $\frac{x^2-3x+6}{(x-1)(3-2x)(x+1)}$ into partial fractions.

Example 2: (case 2) Resolve $\frac{11x - 6 - x^2}{(x+2)(x-2)^2}$ into partial fractions.

Example 3: (case 3) Resolve $\dfrac{x^2 + 2x - 2}{(x-2)(x^2 + x + 1)}$ into partial fractions.

Example 4: (case 4) Resolve $\frac{x^4 - x^2 + 1}{x(x^2 + x - 1)^2}$ into partial fractions.

Before resolving a rational expression into partial fractions always make sure the rational expression is a “proper rational expression” (**numerator is of less degree than the denominator**). If not, divide.

Example 5: Divide so as to produce a “mixed expression” whose fractional part is in proper rational form:

$$\frac{x^3 + 2x^2 - 2x + 2}{x^2 + x - 2}$$

Assignment: Resolve the following rational expressions into partial fractions.

1. $\dfrac{x-13}{(x-3)(x+3)}$

2. $\dfrac{x+2}{2x^2-x}$

3. $\dfrac{x^2}{(x+1)^2(x-1)}$

4. $\dfrac{15-12x}{(x-2)^2(2x-1)^2}$

5. $\dfrac{x^2-x+13}{(x+1)(x^2+4)}$

6. $\dfrac{x^2+x-1}{2(x-1)(x^2-x+1)}$

7. $\dfrac{3x-x^2-2}{(x^2-x+2)^2}$

8. $\dfrac{x^3-3x^2-x+8}{(x-2)^2}$