# Blue Pelican Calculus

# First Semester

Absent-student Version 1.01



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# Calculus AP Syllabus (First Semester)

**Unit 1: Function limits and continuity**

Lesson 01: Limit fundamentals, definitions

Lesson 02: Limits of rational and graphed functions

Lesson 03: Limit theorems, limits of trig functions

Lesson 04: Limits involving infinity

Lesson 05: Piecewise functions and continuity

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Unit 1 test

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Lesson 03: Formal definition of the derivative

Lesson 04: A graphical look at derivatives

Lesson 05: Differentiability

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Unit 2 test

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Cumulative review
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Lesson 3: Derivatives of exponential functions

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**Unit 9: Antiderivatives (Indefinite integrals)**

Lesson 1: Basic integration rules, integrating polynomials

Lesson 2. More integration practice

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Lesson 4: Integration using the chain rule in reverse

Lesson 5: Applications of integration, evaluation of integration constants

Lesson 6: Indefinite integrals with a graphing calculator

Unit 9 review
Unit 9 test

**Semester summary**

Semester review

Semester test

**Enrichment Topics**

**Topic A:** Special sine and cosine limits

**Topic B:** Formal definition of continuity

**Topic C:** Verification of the power rule

**Topic D:** Verification of the product and quotient rules

**Topic E:** Verification of rules for derivative of sine and cosine functions

**Topic F:** Verification of the Chain Rule

**Topic G:** Verification of derivatives of exponential functions

**Topic H:** Verification of derivatives of logarithm functions

**Topic I:** Verification of derivatives of inverse trig functions

**Topic J:** An argument in support of the Fundamental Theorem of Calculus

**Topic k:** Why the absolute value for the integral of 1/x?

**Topic L:** Partial fractions

# Calculus, Unit 1

# Function limits and continuity

# Unit 1: Lesson 01 Limit fundamentals, definitions

Consider

$$\lim_{x \to 2}(x^2 - 5x)$$

Read this as either

"**The limit as *x* goes to two,** of *x* squared minus five *x*."

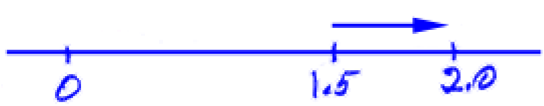
or

"**The limit** of *x* squared minus five *x*, **as *x* goes to two**."

The answer to the above limit can be thought of as the value that the function y = f(x) = $x^2 - 5x$ approaches as *x* gets closer and closer to 2.

Let *f(x)* be a function defined at every number in an open interval containing *a*, except possibly at *a* itself. If the function values of *f(x)* approach a specific number *L* as *x* approaches *a*, then *L is the limit of f(x) as x approaches a*.

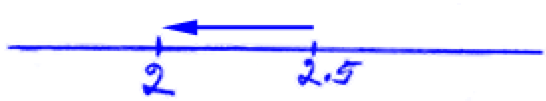
For the function $x^2 - 5x$ above, let *x* approach 2 in a table as follows (Consult **Calculator Appendix AE** and an associated video for how to produce this table on a graphing calculator):




| x | f(x) = $x^2 - 5x$ |
|---|---|
| 1.5 | **–5.25** |
| 1.6 | **–5.44** |
| 1.7 | **–5.61** |
| 1.8 | **–5.76** |
| 1.9 | **–5.89** |
| 2.0 | **–6.0** |

In the table above, the right column (the function value) seems to approach –6 and, in fact, is exactly –6 at x = 2.

For the same function let's approach x = 2 **from the right** now instead of the left.




| x | f(x) = $x^2 - 5x$ |
|---|---|
| 2.5 | **–6.25** |
| 2.4 | **–6.24** |
| 2.3 | **–6.21** |
| 2.2 | **–6.16** |
| 2.1 | **–6.09** |
| 2.0 | **–6.0** |

Again, the limit seems to be approaching –6. Notice that for our function $f(x) = x^2 - 5x$, $f(2) = 2^2 - 5(2) = -6$.

So why use the tables to find what the function value approaches as *x* approaches 2? **Why not just evaluate f(2) and be done with it?**

The fact is, we can do exactly that if the function is a **polynomial**.

If f(x) is a polynomial, then:

$$\lim_{x \to a} f(x) = f(a)$$

**Example 1:** Evaluate

$$\lim_{x \to 3} (2x^3 - x + 1)$$

$f(3) = 2(3)^3 - 3 + 1 = 2(27) - 2 = 54 - 2 = \boxed{52}$

**Approaching from the left:** Consider this table from the previous page. Notice that we are approaching 2 from the left. The notation for this **one sided limit** is:

$\lim_{x \to 2^-} f(x) = -6$

| x | $f(x) = x^2 - 5x$ |
|---|---|
| 1.5 | –5.25 |
| 1.6 | –5.44 |
| 1.7 | –5.61 |
| 1.8 | –5.76 |
| 1.9 | –5.89 |
| 2.0 | –6.0 |

**Approaching from the right:** Consider this table from the previous page. Notice that we are approaching 2 from the right. The notation for this **one sided limit** is:

$\lim_{x \to 2^+} f(x) = -6$

| x | $f(x) = x^2 - 5x$ |
|---|---|
| 2.5 | –6.25 |
| 2.4 | –6.24 |
| 2.3 | –6.21 |
| 2.2 | –6.16 |
| 2.1 | –6.09 |
| 2.0 | –6.0 |

Only when the limits of a function from **both left and right agree** can we say what the limit is in general:

If $\lim\limits_{x\to a^-} f(x) = L$ and $\lim\limits_{x\to a^+} f(x) = L$ then $\mathbf{\lim\limits_{x\to a} f(x) = L}$

In the following two examples, state the general limit in limit notation and the numeric answer (if it exists).

| | |
|---|---|
| **Example 2:**<br>$\lim\limits_{x\to 3^-} f(x) = 11$ and $\lim\limits_{x\to 3^+} f(x) = 11$<br><br>$\lim\limits_{x\to 3} F(x) = 11$ | **Example 3:**<br>$\lim\limits_{x\to 2^-} f(x) = -4$ and $\lim\limits_{x\to 2^+} f(x) = 4$<br><br>$-4 \neq 4$<br>$\lim\limits_{x\to 2} F(x) =$ D.N.E<br>(Does not exist) |

**Assignment:**

1. Write out this limit expression in words.

$$\lim_{x \to -4} (x^3 + 1)$$

2. Convert "The limit of the square root of *x* plus 1, plus *x*, minus 3, as *x* goes to 17" into the mathematical notation for limits.

3. Evaluate

$$\lim_{x \to -4} (x^2 + 8x - 1)$$

4. Evaluate

$$\lim_{x \to 1} (-5x^3 + x^2 + 2)$$

In problems 5 – 8, state the problem in limit notation and what it seems to be approaching. If no apparent limit exists, then so state.

5.

| x | f(x) |
|---|---|
| 4.4 | 13.1 |
| 4.49 | 13.01 |
| 4.499 | 13.001 |
| 4.4999 | 13.0001 |
| 4.49999 | 13.00001 |

6.

| x | f(x) |
|---|---|
| −11.2 | −.1 |
| −11.18 | −.09 |
| −11.10 | −.009 |
| −11.02 | −.004 |
| −11.001 | −.001 |

7.

| x | f(x) |
|---|---|
| 4.4 | 13.1 |
| 4.49 | 13.2 |
| 4.499 | 13.4 |
| 4.4999 | 13.7 |
| 4.49999 | 14.1 |

8.

| x | f(x) |
|---|---|
| 2.2 | 17 |
| 2.1 | 17 |
| 2.01 | 17 |
| 2.001 | 17 |
| 2.0001 | 17 |

9. Write out this limit statement in words.

$$\lim_{x \to a^+} (x^3 + 1) = \mathrm{m}$$

10. Convert “The limit as x approaches b from the left, of f(x).” into mathematical terminology using limit notation.

In problems 11-14, use the two one-sided limits to state the general limit in limit notation and the numeric answer (if it exists).

11.
$\lim_{x \to 0^-} \mathrm{f(x)} = -1$ and $\lim_{\mathrm{x} \to 0^+} \mathrm{f(x)} = -1$

12.
$\lim_{x \to 47^-} \mathrm{f(x)} = 0$ and $\lim_{\mathrm{x} \to 47^+} \mathrm{f(x)} = 0$

13. $f(x) = x^2 - x - 1$

$\lim_{x \to 0^-} f(x) = -1$ & $\lim_{x \to 0^+} f(x) = -1$

14. $f(x) = 1/(x-5)$

$\lim_{x \to 5^-} f(x) = ?$ & $\lim_{x \to 5^+} f(x) = ?$

# Unit 1: Lesson 02 Limits of rational and graphed functions

To find $\lim_{x \to a} f(x)$

- If f(x) is a polynomial, simply evaluate f(a).
- If f(x) is not a polynomial (such as a rational expression), try to evaluate f(a) unless it gives some indeterminate form such as:
  - Division by zero
  - Undefined
  - $\infty/\infty$, etc.

  In these cases, try to algebraically eliminate the difficulty before substituting in the *a* value.

**Example 1:** Find

$$\lim_{x \to 3}\left(\frac{x}{x+2}\right)$$

$$= \frac{3}{3+2} = \boxed{\frac{3}{5}}$$

**Example 2:** Find

$$\lim_{x \to 3}\left(\frac{x^2+2x-15}{x-3}\right)$$

$$= \lim_{x \to 3} \frac{(x+5)\cancel{(x-3)}}{\cancel{x-3}}$$

$$= \lim_{x \to 3} (x+5)$$

$$= 3+5 = \boxed{8}$$

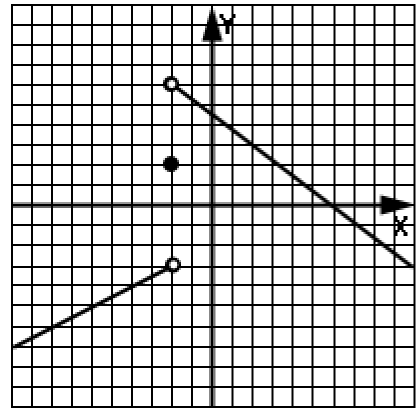
**Example 3:** Find

$$\lim_{x \to 4}\left(\frac{\sqrt{x}-2}{x-4}\right)$$

$$= \lim_{x \to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)} = \lim_{x \to 4} \frac{\cancel{x-4}}{\cancel{(x-4)}(\sqrt{x}+2)}$$

$$= \lim_{x \to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \boxed{\frac{1}{4}}$$

**Example 4:** For f(x) = y, find




$\lim_{x \to -2^-} f(x) = \mathbf{-3}$

These two don't agree

$\lim_{x \to -2^+} f(x) = \mathbf{6}$

$\lim_{x \to -2} f(x) = \textbf{D. N. E. (does not exist)}$

$f(-2) = \mathbf{2}$

**Assignment:** Find the indicated limits.

1. $\lim_{x\to -2}\dfrac{x^2-5x-14}{x+2}$

2. $\lim_{x\to 6}(x^2+x-2)$

3. $\lim_{x\to 5}\dfrac{\sqrt{x}-\sqrt{5}}{x-5}$

4. $\lim_{x\to 1}\dfrac{x-1}{x^2-4x+3}$

5. $\lim_{x\to 4}\dfrac{5x-20}{x^2-16}$

6. $\lim_{x\to 4}\dfrac{\sqrt{x}-2}{x+4}$

7. $\lim_{x\to 2}\frac{x^4-16}{x^2-4}$

8. $\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}$

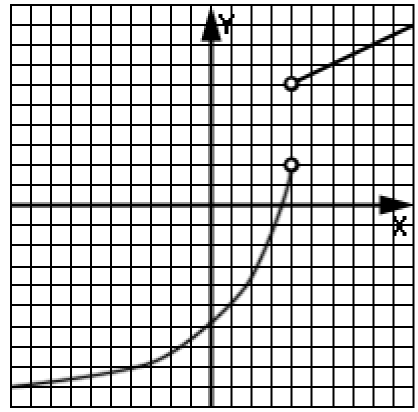
9. $\lim_{x\to 0}(x^2-5x-1)$

10. $\lim_{x\to 2}\frac{|x+2|}{x+2}$

11. $\lim_{x\to 5}\frac{5-x}{\sqrt{x}-\sqrt{5}}$

12. $\lim_{x\to 9}\frac{9-x}{81-x^2}$

13. Find the following for f(x) = y




$\lim_{x \to 4^-} f(x) =$

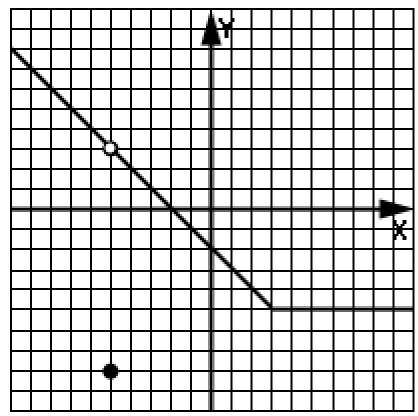
$\lim_{x \to 4^+} f(x) =$

$\lim_{x \to 4} f(x) =$

$\lim_{x \to 6} f(x) =$

$f(4) =$

14. Find the following for f(x) = y




$\lim_{x \to -5^-} f(x) =$

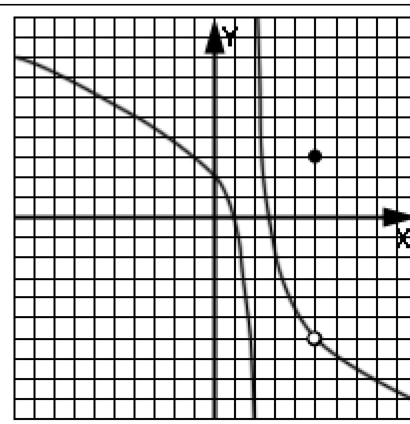
$\lim_{x \to -5^+} f(x) =$

$\lim_{x \to -5} f(x) =$

$\lim_{x \to 3} f(x) =$

$f(-5) =$

15. Find the following for f(x) = y




$\lim_{x \to 2^-} f(x) =$

$\lim_{x \to 2^+} f(x) =$

$\lim_{x \to 5} f(x) =$

$f(2) =$

$f(5) =$

16. Sketch the function, $f(x) = \sqrt{x-3} + 2$. Use the sketch to find the following limits.

$\lim\limits_{x \to 3^-} f(x) =$

$\lim\limits_{x \to 3^+} f(x) =$

$\lim\limits_{x \to 3} f(x) =$

$\lim\limits_{x \to 12} f(x) =$

## Unit 1: Lesson 03 Limit theorems, limits of trig functions

For the following limit theorems, assume:

$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = M$$

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$$

$$\lim_{x\to a}[f(x)g(x)] = [\lim_{x\to a} f(x)] \cdot \left[\lim_{x\to a} g(x)\right] = L \cdot M$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M} \qquad \lim_{x\to a}(k \cdot f(x)) = k \lim_{x\to a} f(x) = k\,L$$

(where k is a constant)

**Example 1:** Assume $\lim_{x\to 3} f(x) = -1$ and $\lim_{x\to 3} g(x) = 7$

$$\lim_{x\to 3}[3g(x) - f(x)] = ?$$

$$= \lim_{x\to 3} 3g(x) - \lim_{x\to 3} f(x) = 3\overset{7}{\lim_{x\to 3} g(x)} - \overset{-1}{\lim_{x\to 3} f(x)}$$

$$= 3\cdot 7 + 1 = 21 + 1 = \boxed{22}$$

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$$\lim_{x\to 3} \frac{x + f(x)}{g(x) - f(x)}$$

$$= \lim_{x\to 3}[x + f(x)] \Big/ \lim_{x\to 3}[g(x) - f(x)]$$

$$= \left[3 + \lim_{x\to 3} f(x)\right] \Big/ \left[\lim_{x\to 3} g(x) - \lim_{x\to 3} f(x)\right]$$

$$= [3 + (-1)] / [7 - (-1)] = 2/8 = \boxed{\tfrac{1}{4}}$$

The following special trig limits should be memorized (See **Enrichment Topic A** for their justification):

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{x}{\sin(x)} = 1$$

$$\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$$

The following trig approximations are useful as ***x* (in radians) approaches 0**.

$\sin(x) \approx x$ … comes from $\sin(x) = x - x^3/3! + x^5/5! - x^7/7! + \ldots$

$\cos(x) \approx 1$ … comes from $\cos(x) = 1 - x^2/2! + x^4/4! - x^6/6! + \ldots$

In finding the limits of trig functions, use direct substitution first. If that yields an indeterminate form, then use one of the special cases above.

**Example 2:** $\lim_{\theta\to 0} \frac{\sin(8\theta)}{\theta} = ?$

$$= \lim_{\theta\to 0} \frac{8\sin(8\theta)}{8\theta} = \lim_{\theta\to 0} 8\,\underbrace{\cancel{\frac{\sin(8\theta)}{8\theta}}}_{\to 1}$$

$$= \boxed{8}$$

**Example 3:** $\lim_{\alpha\to 0} \tan(\alpha) = ?$

Use direct substitution:

$= \tan(0) = \boxed{0}$

**Assignment:** For problems 1- 4, assume the following:

$$\lim_{x \to -4} f(x) = 1 \quad \text{and} \quad \lim_{x \to -4} g(x) = -2$$

1. $\displaystyle\lim_{x \to -4} \frac{g(x)}{f(x) + x}$

2. $\displaystyle\lim_{x \to -4} [xf(x) - g(x)]$

3. $\displaystyle\lim_{x \to -4} [f(x)^2 - g(x)^2 - 2]$

4. $\displaystyle\lim_{x \to -4} [f(x) + g(x)]^2$

5. $\displaystyle\lim_{x \to 0} \frac{\tan(x)}{x} = ?$

6. $\lim\limits_{x \to 0} \dfrac{-\sin(\pi\, x)}{\pi\, x} = ?$

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7. $\lim\limits_{\theta \to 0} \dfrac{\cos^2(\theta) - 1}{\theta} = ?$

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8. $\lim\limits_{x \to \pi} \cos(x) = ?$

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9. $\lim\limits_{b \to 0} \dfrac{(1 - \cos(b))^2}{b} = ?$

10. $\lim\limits_{x \to 0} \dfrac{x}{\sin(7x)} = ?$

11. $\lim\limits_{\theta \to \pi/2} \dfrac{\cot(\theta)}{\cos(\theta)} = ?$

12. $\lim\limits_{x \to 5} \tan\left(\dfrac{\pi x}{4}\right) = ?$

13. $\lim\limits_{\beta \to 0} \dfrac{\sin(\beta)\cos(\beta)}{\beta}$

14. $\lim_{x \to 9} \dfrac{18 - 2x}{3 - \sqrt{x}} = ?$

# Unit 1: Lesson 04 Limits involving infinity

A fundamental limit on which many others depend is:

$$\lim_{x\to\infty}\frac{1}{x^n}=0 \quad ; \text{n is any positive power}$$

| x | f(x) = 1/x[1] |
|---|---|
| 100 | .01 |
| 1,000 | .001 |
| 10,000 | .0001 |
| 100,000 | .00001 |
| 1,000,000 | .000001 |

Infinity (∞) is **not a position on the number line.** Rather, it is a **concept** of a number continuing to get larger and larger without any limit. With that in mind, consider the problem:

$$\lim_{x\to\infty}\frac{3x^2}{x^2+5}$$

What happens if we try to "substitute in ∞" (which is illegal since ∞ is not a number)? We would illegally obtain the following:

$$\frac{3\infty^2}{\infty^2+5}=\frac{\infty}{\infty}$$

Can we just cancel ∞/∞ to make 1? No, because ∞ is not a number that could be canceled as could be done with 5/5. Example 1 below shows the proper way to handle this problem where the answer will be shown to be 3, not 1.

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**Example 1:** $\lim_{x\to\infty}\frac{3x^2}{x^2+5}=?$

$$\lim_{x\to\infty}\frac{3x^2}{x^2+5}\cdot\frac{1/x^2}{1/x^2}=\lim_{x\to\infty}\frac{\frac{3x^2}{x^2}}{\frac{x^2}{x^2}+\frac{5}{x^2}}$$

$$=\lim_{x\to\infty}\frac{3}{1+5/x^2}=\frac{3}{1+0}=\boxed{3}$$

As a general rule in handling a problem such as Example 1, find the **highest degree** in both the numerator and denominator (assume it's *n*) and multiply by 1 in this form:

$$\frac{\frac{1}{x^n}}{\frac{1}{x^n}}$$

**Example 2:** $\lim\limits_{x\to\infty} \frac{7x^2-2x}{4x^3-x} = ?$

$$\lim_{x\to\infty} \frac{7x^2-2x}{4x^3-x} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{7x^2}{x^3}-\frac{2x}{x^3}}{\frac{4x^3}{x^3}-\frac{x}{x^3}}$$

$$= \lim_{x\to\infty} \frac{\frac{7}{x}-\frac{2}{x^2}}{4-\frac{1}{x^2}} = \frac{0-0}{4-0} = \frac{0}{4} = \boxed{0}$$

**Example 3:** $\lim\limits_{x\to\infty}(x^3-6x^2+x) = ?$

Factor out $x^3$

$$\lim_{x\to\infty}\left[x^3\left(1-\frac{6}{x}+\frac{1}{x^2}\right)\right]$$

$$=\left[\lim_{x\to\infty} x^3\right]\left[\lim_{x\to\infty}\left(1-\frac{6}{x}+\frac{1}{x^2}\right)\right]$$

$$=(+\infty)[1-0+0] = +\infty(1) = \boxed{+\infty}$$

Notice in Example 3 that the other terms pale in comparison to $x^3$ as $x$ goes to infinity. Therefore, we have the following rule:

For any polynomial, P(x)

$$\lim_{x\to\pm\infty} P(x) = \lim_{x\to\pm\infty} (\text{highest power term of } P(x))$$

**Example 4:** $\lim\limits_{x\to-\infty}(11x^2 - 2x^3 + x) = ?$

Factor out $x^3$

$= \lim\limits_{x\to-\infty}\left[x^3\left(\frac{11}{x} - 2 + \frac{1}{x^2}\right)\right]$

$= \left[\lim\limits_{x\to-\infty}(x^3)\right]\left[\lim\limits_{x\to-\infty}\left(\frac{11}{x} - 2 + \frac{1}{x^2}\right)\right]$

$= (-\infty)[0 - 2 + 0] = (-\infty)(-2) = \boxed{+\infty}$

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Consider the problem $\lim\limits_{x\to2^-}\dfrac{2x^2-5x+1}{x^2+x-6}$

Direct substitution of x = 2 yields: $\dfrac{-1}{0}$

So is the answer +∞ or –∞? Example 5 shows the correct way to analyze this problem.

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**Example 5:** $\lim\limits_{x\to2^-}\dfrac{2x^2-5x+1}{x^2+x-6} = ?$

$= \lim\limits_{x\to2^-}\dfrac{2x^2-5x+1}{(x-2)(x+3)} = \lim\limits_{x\to2^-}\left(\dfrac{1}{x-2}\right)\left[\lim\limits_{x\to2^-}\dfrac{2x^2-5x+1}{x+3}\right]$

$= [-\infty]\left[(2(2)^2-5(2)+1)/(2+3)\right]$

$= [-\infty]\ \dfrac{[8-10+1]}{5} = [-\infty]\ [-1/5]$

$= \boxed{+\infty}$

**Assignment:**

1. $\lim_{x \to \infty} \frac{x+5}{x-2} = ?$

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2. $\lim_{x \to \infty} \frac{5x^3+2}{20x^3-6x}$

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3. $\lim_{x \to \infty} \frac{5+2^x}{15-6x}$

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4. $\lim_{x \to \infty} (7 - 11x^2 - 6x^5)$

5. $\lim_{x\to\infty} 6^x = ?$

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6. $\lim_{x\to\infty} \frac{9x^4 - x^3 + 1}{x - 2x^4}$

---

7. $\lim_{x\to-\infty} (12x^4 - x^3 + 7x^2 + 1)$

---

8. $\lim_{x\to\infty} \left(7 - \frac{1}{x} + \frac{1}{x^2}\right)$

9. $\lim\limits_{x\to 1^+} \dfrac{3x^4 - x + 1}{x^2 - 6x + 5}$

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10. $\lim\limits_{x\to -\infty} (x^3 - 2{,}000{,}000)$

---

11. $\lim\limits_{x\to 1^-} \dfrac{x^2 - 2x + 1}{x^3 - 3x^2 + 3x - 1}$

---

12. $\lim\limits_{x\to 4^-} \dfrac{x^{50} - 3x^{49}}{x - 4}$

# Unit 1: Lesson 05 Piecewise functions and continuity

A function is discontinuous at a particular *x* value if we need to "**lift the pencil"** at that point in order to keep drawing that function. Otherwise, the function is said to be continuous there. See **Enrichment Topic B** for a more formal definition of discontinuity.

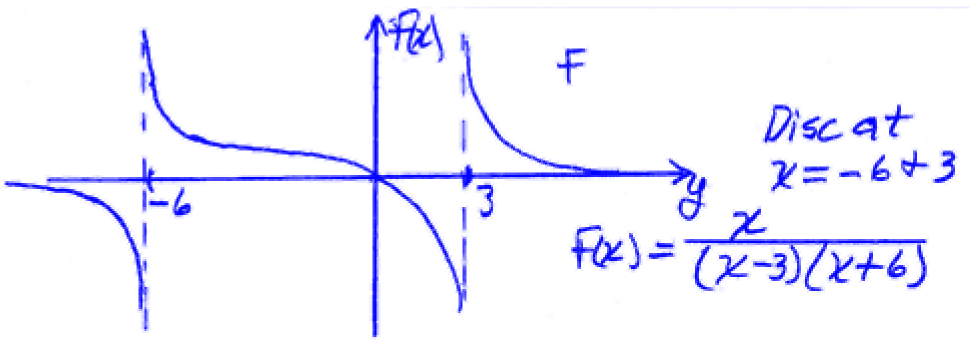
There are several things that can cause a discontinuity at x = a for a function:

- There is a vertical asymptote at x = a . Typically, (x – a) is a factor of the denominator. (See Example 1).
- A piecewise function abruptly "jumps" at x = a. (See Example 3.)
- There is a "hole" in the graph at x = a. (See Example 5.)

**Polynomials are continuous everywhere.**

**Example 1:** Sketch the graph of f(x) (and note the positions of any discontinuities).

$$f(x) = \frac{x}{x^2 + 3x - 18}$$




**Example 2:** Just by observing the sketch in Example 1, determine the following limits:

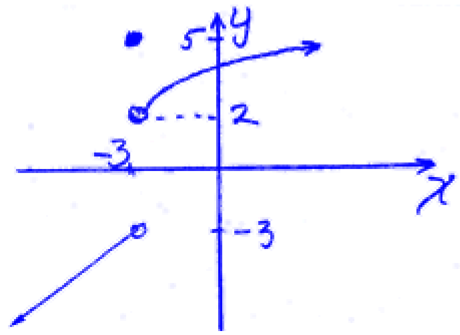
$\lim_{x \to -6^-} f(x) = -\infty$ $\qquad$ $\lim_{x \to -6^+} f(x) = +\infty$

$\lim_{x \to 3^-} f(x) = -\infty$ $\qquad$ $\lim_{x \to 3^+} f(x) = +\infty$

$\lim_{x \to 3} f(x) = \mathbf{D.N.E.}$ $\qquad$ $f(3) = \mathbf{D.N.E.}$

**Example 3:** Sketch this piecewise function.

$$f(x) = \begin{cases} x & when\ x < -3 \\ 5 & when\ x = -3 \\ \sqrt{x+3}+2 & when\ x > -3 \end{cases}$$

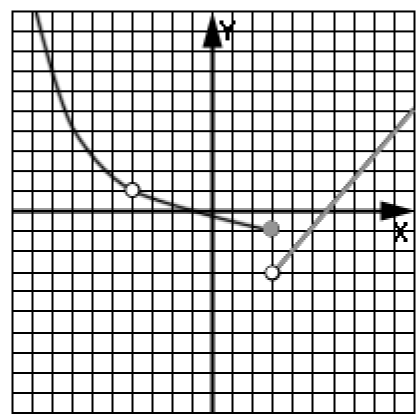



**Example 4:** Just by observing the sketch in Example 3, determine the following values:

$\lim_{x\to -3^-} f(x) = \mathbf{-3}$  $\lim_{x\to -3^+} f(x) = \mathbf{2}$

$\lim_{x\to -3} f(x) = \mathbf{D.N.E.}$  $f(-3) = \mathbf{5}$

**Example 5:** State the *x* positions of discontinuity and identify which are "holes."




**x = –4 (hole) and x = 3**

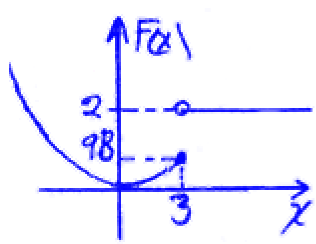
$\lim_{x\to -4^-} f(x) = \mathbf{1}$  $\lim_{x\to -4^+} f(x) = \mathbf{1}$

$\lim_{x\to -4} f(x) = \mathbf{1}$  $\lim_{x\to 3^-} f(x) = \mathbf{-1}$

$\lim_{x\to 3^+} f(x) = \mathbf{-3}$  $\lim_{x\to 3} f(x) = \mathbf{D.N.E.}$

**Example 6:** Determine the value of B so as to insure that this function is everywhere continuous.

$$f(x) = \begin{cases} Bx^2 & \text{if } x \le 3 \\ 2 & \text{if } x > 3 \end{cases}$$




$f(3) = B\,3^2$
$= 9B$
$9B = 2$
$B = \boxed{\frac{2}{9}}$

**Assignment:** In problems 1-3, sketch the function and identify any positions of discontinuity.

1. $$f(x) = \begin{cases} \dfrac{x^4 - 81}{x - 3} & \text{if } x \neq 3 \\ 9 & \text{if } x = 3 \end{cases}$$

$\lim_{x \to 3^-} f(x) =$

$\lim_{x \to 3^+} f(x) =$

$\lim_{x \to 3} f(x) =$

---

2. $$f(x) = 4 - \frac{1}{x} + \frac{x^2}{x - 5}$$

$\lim_{x \to 0^-} f(x) =$

$\lim_{x \to 5^+} f(x) =$

---

3. $$f(x) = \begin{cases} \dfrac{x^2 - 6x}{20x - 120} & \text{if } x \neq 6 \\ \dfrac{3}{10} & \text{if } x = 6 \end{cases}$$

$\lim_{x \to 6^-} f(x) =$

$\lim_{x \to 6^+} f(x) =$

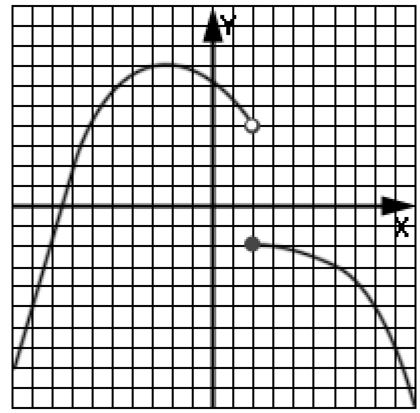
$\lim_{x \to 6} f(x) =$

---

4. Algebraically "design" a linear function that has a hole at x = 2, but whose limit as x approaches 2 is 5.

In problems 5-7, state the *x* positions of discontinuity and answer the questions.

5.



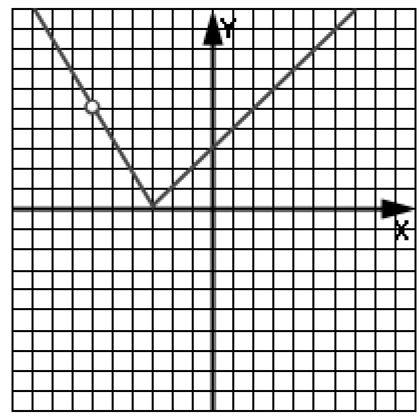

$\lim\limits_{x\to 2^-} f(x) =$

$\lim\limits_{x\to 2^+} f(x) =$

$\lim\limits_{x\to 2} f(x) =$

$f(2) =$

6.



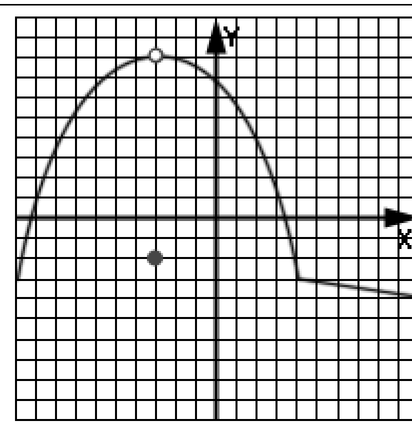

$\lim\limits_{x\to -6^-} f(x) =$

$\lim\limits_{x\to -6^+} f(x) =$

$\lim\limits_{x\to -6} f(x) =$

$f(-6) =$

7.




$\lim\limits_{x\to -3^-} f(x) =$

$\lim\limits_{x\to -3^+} f(x) =$

$\lim\limits_{x\to -3} f(x) =$

$f(-3) =$

8. State the position of discontinuity of $f(x) = 8x^4 - 3x^3 + x^2 - 6$

9. Determine the value of *b* so as to insure that the function is everywhere continuous.

$$f(x) = \begin{Bmatrix} 3x + b & \text{if } x \leq 2 \\ -x - 1 & \text{if } x > 2 \end{Bmatrix}$$

---

10. Determine the values of *m* and *b* so as to insure that the function is everywhere continuous.

$$f(x) = \begin{Bmatrix} 4 & if\ x \leq 3 \\ mx + b & \text{if } 3 < x < 6 \\ 1 & if\ x \geq 6 \end{Bmatrix}$$

# Unit 1: Review

1. Write out this limit expression in words:

$$\lim_{x\to -5^+} f(x)$$

---

In problems 2 and 3, state the problem in (one-sided) limit notation and what it seems to be approaching. If no apparent limit exists, then so state.

2.

| x | f(x) |
|---|---|
| 5.75 | 500 |
| 5.71 | 1002 |
| 5.7001 | 100,005 |
| 5.70002 | 2,000,500 |
| 5.700009 | 120,010,075 |

3.

| x | f(x) |
|---|---|
| –6.12 | π/3 |
| –6.11 | π/100 |
| –6.103 | π/1000 |
| –6.10054 | π/100,000 |
| –6.100003 | π/1,000,000 |

---

In problems 4 and 5, give the general limit (if it exists).

4. $f(x) = x^2 - 4x - 1$

$$\lim_{x\to 2} f(x) = ?$$

5. $f(x) = 1/(x + 8)$

$$\lim_{x\to -8} f(x) = ?$$

---

6. $$\lim_{x\to 6} \frac{x^2 + 4x - 12}{x + 6} = ?$$

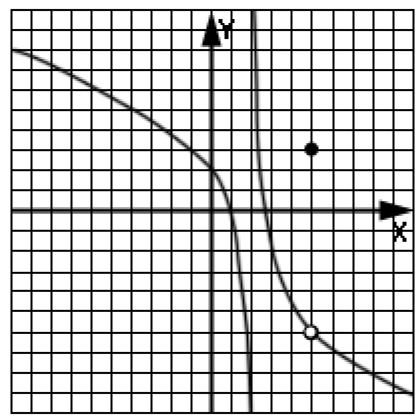
7. $\lim_{x \to 7} \frac{\sqrt{x} - \sqrt{7}}{x - 7} = ?$

---

8. $\lim_{x \to 8} \frac{x}{|x + 8|}$

---

9. Find the following for f(x) = y




$\lim_{x \to 2^-} f(x) = ?$

$\lim_{x \to 2^+} f(x) = ?$

$\lim_{x \to 5} f(x) = ?$

$f(2) = ?$

$f(5) = ?$

---

10. Assume $\lim_{x \to -2} f(x) = -6$ and $\lim_{x \to -2} g(x) = 7$

$\lim_{x \to -2} \frac{x + f(x)}{g(x) - f(x)} = ?$

11. Sketch the function, $f(x) = \sqrt{25 - x^2} + 2$. Use the sketch to find the following limits.

$\lim_{x \to 5^-} f(x) = ?$

$\lim_{x \to 5^+} f(x) = ?$

$\lim_{x \to 5} f(x) = ?$

$\lim_{x \to 0} f(x) = ?$

---

12. $\lim_{x \to 0} \frac{cos^2(2x) - 1}{x} = ?$

---

13. $\lim_{x \to 0} \frac{5x}{sin(x)} = ?$

---

14. $\lim_{x \to \infty} \frac{7x^3 - 2x}{4x^3 - x} = ?$

15. $\lim\limits_{x \to -\infty} 3\,\dfrac{5 + 2^x}{15 - 16x} = ?$

16. $\lim\limits_{x \to -\infty} (4 - 10x^2 - 6x^3)$

17. At what *x* value(s) is this function discontinuous?

$$f(x) = \frac{x - 3}{x(x + 9)^2}$$

18. Sketch this piecewise function and then answer the questions.

$$f(x) = \begin{Bmatrix} x & when\ x < -3 \\ .5 & when\ x = -3 \\ -\sqrt{x+3} - 1 & when\ x > -3 \end{Bmatrix}$$

$\lim\limits_{x \to -3^-} f(x) = ?$ $\lim\limits_{x \to -3^+} f(x) = ?$

$\lim\limits_{x \to -3} f(x) = ?$ $f(-3) = ?$

19. Determine the value of C so as to insure that this function is everywhere continuous.

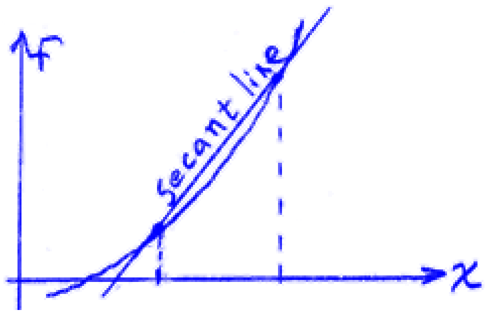
$$f(x) = \begin{Bmatrix} Cx^3 & \text{if } x \le 1 \\ -2 & \text{if } x > 1 \end{Bmatrix}$$

# Calculus, Unit 2

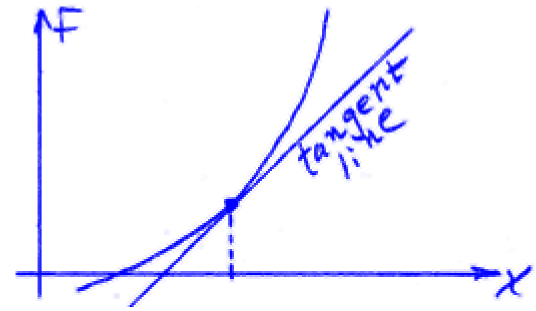
# Derivative fundamentals

# Unit 2: Lesson 01 — Average and instantaneous rates of change
## Definition of the derivative at x = c

The **average rate of change** between two points on a function is the **slope** of a secant line drawn between those two points.




The **instantaneous rate of change** of a function at a point on that function is the **slope** of a tangent to the curve at that point.

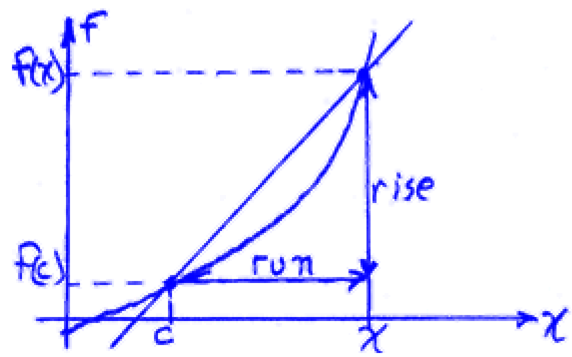



The instantaneous rate of change of a function at a point is called the **derivative** of the function at that point and is defined as a limit:

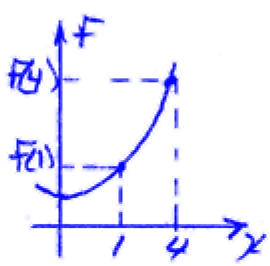
$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Read *f'(c)* as, "*f* prime of c" which means the "derivative of *f evaluated at c*."

$$m = \frac{rise}{run} = \frac{f(x) - f(c)}{x - c}$$




**Example 1:** Find the average rate of change of the function f(x) = $3x^2 + 2$ between x =1 and x =4.




$$m = \frac{f(4)-f(1)}{4-1} = \frac{3\cdot 4^2+2-(3\cdot 1^2+2)}{3}$$

$$= \frac{48+2-(3+2)}{3} = \frac{50-5}{3} = \frac{45}{3}$$

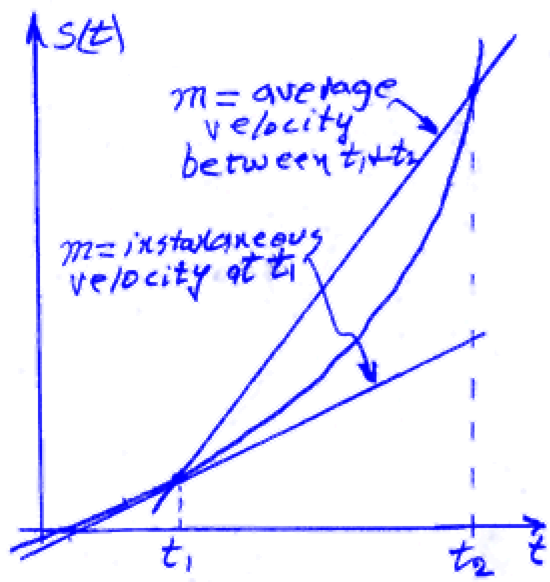
$$= \boxed{15}$$

**Example 2:** Find the instantaneous rate of change of the function $f(x) = 3x^2 + 2$ at $x = c = 1$.

$$f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{3x^2+2-(3\cdot 1^2+2)}{x-1}$$
$$= \lim_{x\to 1} \frac{3x^2+\cancel{2}-3\cdot 1^2-\cancel{2}}{x-1} = \lim_{x\to 1} \frac{3(x^2-1^2)}{x-1}$$
$$= \lim_{x\to 1} \frac{3\cancel{(x-1)}(x+1)}{\cancel{x-1}} = \lim_{x\to 1} 3(x+1) = 3(1+1) = \boxed{6}$$

Are these rates of change (both average and instantaneous) just mathematical abstractions, or are there "real world" applications?

If *s(t)* is the time-position of an object moving along a straight line, then the average rate of change of this function is the **average velocity** (over some time-interval) and the instantaneous rate of change is the **instantaneous velocity** at some particular time.




**Example 3:** Find the average velocity of the object whose time-position is given by $s(t) = t^2 - 6t - 3$ meters, between t = 2 sec and t = 6 sec.

$$V_{av} = m = \frac{s(6)-s(2)}{6-2} = \frac{6^2-6(6)-3-(2^2-6\cdot 2-3)}{4}$$
$$= \frac{\cancel{36}-\cancel{36}-3-(4-12-3)}{4}$$
$$= \frac{-3-(-11)}{4} = \frac{-3+11}{4} = \frac{8}{4} = \boxed{2\text{ m/sec}}$$

**Example 4:** Find the instantaneous velocity of the object whose time-position is given by s(t) = $t^2 - 6t - 3$ meters , at t = 2 sec.

$$s'(c) = \lim_{t\to 2}\frac{s(t)-s(2)}{t-2} = \lim_{t\to 2}\frac{t^2-6t-3-(2^2-6\cdot 2-3)}{t-2}$$

$$= \lim_{t\to 2}\frac{t^2-6t-3-(4-12-3)}{t-2} = \lim_{t\to 2}\frac{t^2-6t-3+11}{t-2}$$

$$= \lim_{t\to 2}\frac{t^2-6t+8}{t-2} = \lim_{t\to 2}\frac{(t-4)\cancel{(t-2)}}{\cancel{t-2}} = \lim_{t\to 2}(t-4)$$

$$= 2-4 = \boxed{-2\ m/sec}$$

**Assignment:**

1. For the function $f(x) = 4x^2$, find the average rate of change between $x = 2$ and $x = 7$.

---

2. For the function $f(x) = 4x^2$, find the instantaneous rate of change at $x = 2$.

---

3. For the time-position function $s(t) = t^3 + 4$ meters, find the average velocity between $t = 3$sec and $t = 11$sec.

4. For the time-position function $s(t) = t^3 + 4$ feet, find the instantaneous velocity at $t = 3$ min.

5. What is the derivative of $f(x) = -2x + 5$ at $x = c = 11$?

6. Find $f'(-2)$ where $f(x) = -5x^2 + x - 12$.

7. Find the average rate of change of the function given by $g(x) = x^3 - x$ over the interval from $x = -1$ to $x = 7$.

---

8. What is the instantaneous velocity of an object in free-fall when its vertical position is given by $s(t) = 400 - 4.9t^2$ meters after $t = 3$ seconds?

---

9. What is $f'(c)$ when $c = 5$ and $f(x) = x^2 - 7x + 2$?

10. What is the slope of the tangent line at $x = -4$ of the curve given by $f(x) = 4x - x^2$?

11. Draw the curve $f(x) = x^2$ and label all that would be necessary to find the slope of the secant line between the two points on the curve given by $x = 1$ and $x = 4$.

## Unit 2: Lesson 02 Equations of tangent and normal lines

In this lesson we will find the equation of the tangent line to a curve at a particular point and also the equation of a normal (perpendicular) line at the point. To do this, use the following:

- The y-value of the point is obtained by **evaluating the function** at the given x-value.
- The **slope** of the tangent line is the **derivative** of the function at that particular x-value.
- The slope of the normal line is the **negative reciprocal** of the slope of the tangent line

**Example 1:** Find the equation of the tangent line to the curve f(x) at x = 3 where $f(x) = 4x^2 – x + 7$.

$$f(3) = 4\cdot 3^2 - 3 + 7 = 36 + 4 = 40$$

$$(x, y) = (3, 40)$$

$$m = f'(3) = \lim_{x\to 3} \frac{f(x) - f(3)}{x-3} = \lim_{x\to 3}\frac{4x^2 - x + 7 - (36 - 3 + 7)}{x-3}$$

$$= \lim_{x\to 3}\frac{4x^2 - x - 33}{x-3} = \lim_{x\to 3}\frac{(4x+11)\cancel{(x-3)}}{\cancel{x-3}}$$

$$= 4\cdot 3 + 11 = 23$$

$y = mx + b$ sub in $(3, 40)$

$y = 23x + b$

$40 = 23\cdot 3 + b$

$-29 = b$

$y = mx + b$

$\boxed{y = 23x - 29}$

**Example 2:** Find the equation of the normal line to the curve f(x) at x = 3 where $f(x) = 4x^2 – x + 7$.

$m_{tan} = 23$

$m_{\perp} = -\frac{1}{23}$

$y = mx + b$

$y = -\frac{1}{23}x + b$

sub in $(3, 40)$

$40 = -\frac{3}{23} + b$

$\frac{23}{23}\cdot\frac{40}{1} + \frac{3}{23} = b$

$\frac{923}{23} = b$

$y = mx + b$

$y = \boxed{-\frac{1}{23}x + \frac{923}{23}}$

**Assignment:**

1. Find the equation of the tangent line to the curve f(x) at $x = -4$ where $f(x) = x^2 - x + 1$.

2. Find the equation of the normal line to the curve f(x) at $x = -4$ where $f(x) = x^2 - x + 1$.

3. What is the equation of the normal line to the curve given by $f(x) = 2/x$ at $x = -1$?

4. What is the equation of the tangent line to the curve given by $f(x) = \sqrt{x}$ at x = 5?

5. Find the equation of the normal line to the curve $x^2/3 + 2$ at the point (3, 5).

6. Sketch the graph of $y = -x^2 + 5$. Without doing any mathematics and just by looking at the sketch, what would you guess the equation of the tangent at x = 0 to be?

7. If $m$ is the slope of the tangent line to the curve given by $f(x) = -x^2$, show that m = –8 at (4, –16).

---

8. If the derivative of f(x) at x = c = 2 is –5 and f(2) = 13, what is the equation of the tangent line at x = 2?

---

9. Consider a parabola having its vertex at (2,1) and passing through (–4, 7). What is the equation of the tangent line at x = 8?

10. What is the equation of the normal line that passes through the vertex of the parabola described in problem 9?

**Unit 2: Lesson 03**

## Formal definition of the derivative

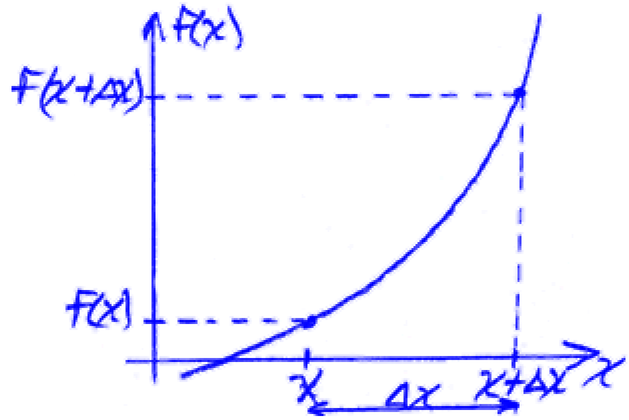
In lesson 1 of this unit, we looked at the definition of the instantaneous rate of change (the derivative) of a function at the **specific** point given by x = c.

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

We now present the **general** formula for the derivative of y = f(x) at the **general** position *x* and its accompanying diagram. Notice the use of *Δx* which means, "the change in *x*."

$$\boldsymbol{f'(x) = y' = \frac{dy}{dx}}$$

$$= \lim_{\Delta x \to 0} \frac{\mathbf{f(x + \Delta x) - f(x)}}{\mathbf{\Delta x}}$$




Notice a new notation for the derivative, $\frac{dy}{dx}$. (Quite often the above formula uses *h* instead of *Δx*).

**Example 1:** Using the formula above find f'(x) where f(x) = $3x^2 - x$.

$$f' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3(x+\Delta x)^2 - (x+\Delta x) - (3x^2 - x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3(x^2 + 2x(\Delta x) + (\Delta x)^2) - \cancel{x} - \Delta x - 3x^2 + \cancel{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{3x^2} + 6x(\Delta x) + 3(\Delta x)^2 - \Delta x - \cancel{3x^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{(\Delta x)}(6x + 3\Delta x - 1)}{\cancel{\Delta x}} = \lim_{\Delta x \to 0} (6x + 3\Delta x - 1) = \boxed{6x - 1}$$

**Example 2:** Use the formal definition of the derivative to find the slope of the tangent line to the curve given by f(x) = $x^2$ + 6x – 2 at x = –4.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + 6(x+\Delta x) - 2 - (x^2+6x-2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x^2} + 2x(\Delta x) + (\Delta x)^2 + \cancel{6x} + 6(\Delta x) - \cancel{2} - \cancel{x^2} - \cancel{6x} + \cancel{2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(2x + \Delta x + 6)}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} \frac{(2x+\Delta x+6)}{1} = 2x + 6$$

$$f'(-4) = 2(-4) + 6 = \boxed{-2} = m$$

**Assignment:** In the following problems, **use the new formal definition** to find the derivative of the function and **then** substitute in a particular value if asked to do so.

1. If $y = f(x) = x^2$, find $\frac{dy}{dx}$.

---

2. What is $f'(x)$ where $f(x) = (x - 5)/4$ ?

---

3. Evaluate $y'$ at $x = 17$ where $y = 7x^2 + 2x - 1$.

4. What is the slope of the normal line to the curve given by $f(x) = \sqrt{x}$ at x = 1?

5. $\lim_{x \to 3} \dfrac{\sqrt{x+1} - 2}{x - 3} = ?$

6. What is the slope of the tangent line to the curve given by f(x) = 1/x at x = 6?

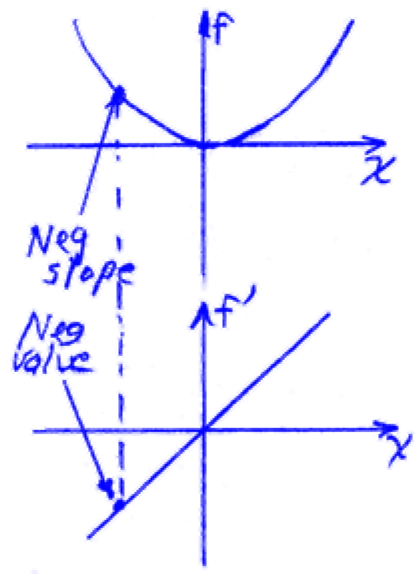
7. Find the equation of the tangent line to the curve $f(x) = x^3$ at $x = -5$.

**Unit 2: Lesson 04** **A graphical look at derivatives**

Recall that the derivative of a function at a point is really the **slope** of a tangent line at that point.

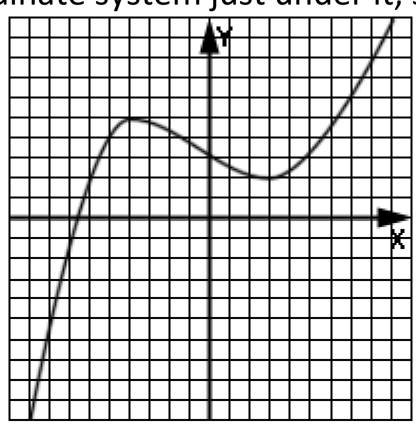
Graph both $f(x) = x^2$ and its derivative $f'(x) = 2x$ in the space provided to the right. Notice that at each corresponding x-value, *f′* is the **slope** of *f*.

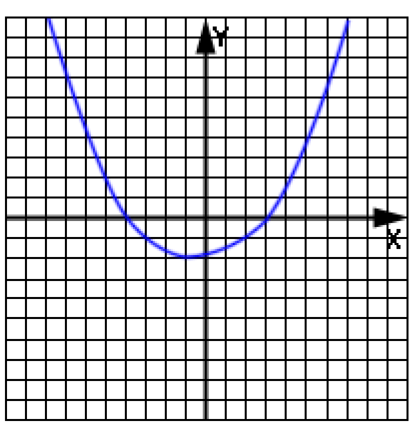
It is generally true of all polynomials, that the **degree** of the derivative *f′* is **one less** than that of the original function *f*.



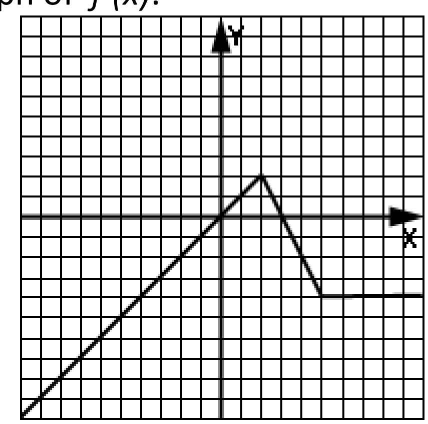

In the following two examples, consider the top graph as the original function *f(x)*. On the coordinate system just under it, sketch the graph of *f′(x)*.

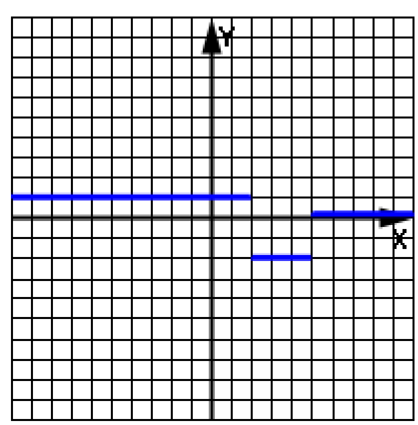
**Example 1:**



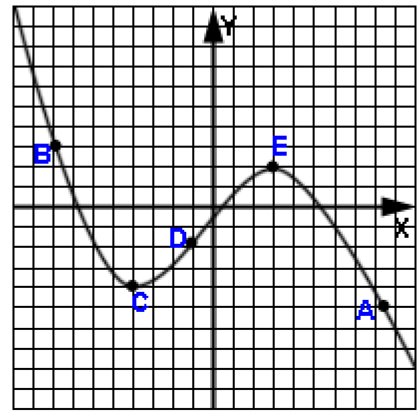

**Example 2:**



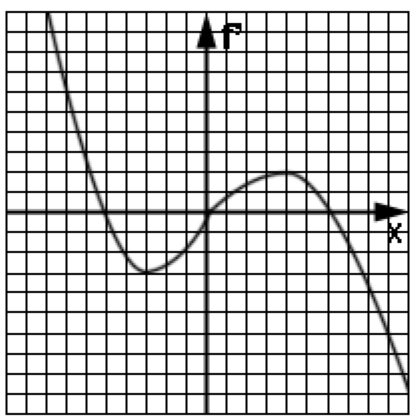



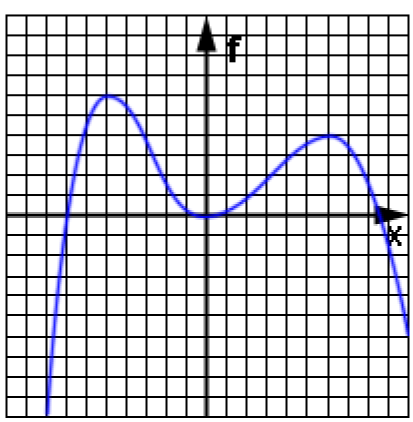
**Example 3:** Label directly on the graph the points as described below.

A. Point A has a negative derivative and a negative function value.
B. Point B has a positive function value and a negative derivative.
C. The slope of the tangent line at point C is 0 and the function value is negative.
D. Point D has a positive derivative.
E. Point E is a maximum point in its own little "neighborhood" and has a positive function value.

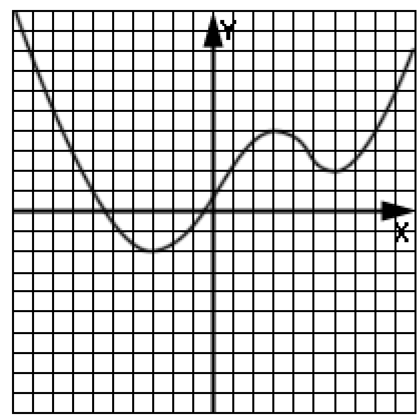



**Example 4:** Given that the top graph is the derivative *f′*, sketch the original function *f* on the bottom coordinate system.




**Example 5:** Identify the requested intervals for the function shown here.




a. Interval(s) of negative derivative
**$(-\infty,-3), (3, 6)$**

b. Interval(s) of function decrease
**$(-\infty,-3), (3, 6)$**

c. Interval(s) of positive derivative
**$(-3, 3), (6, \infty)$**

d. Interval(s) of function increase
**$(-3, 3), (6, \infty)$**

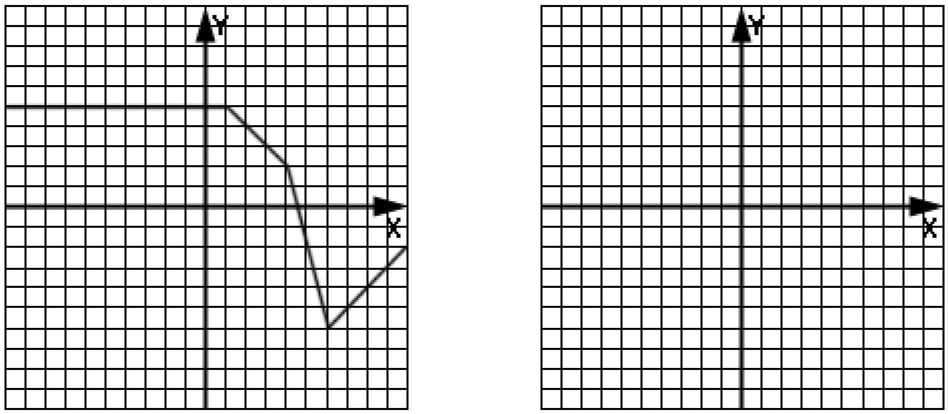
Notice from example 5 that we can infer the following:

- Intervals of **negative** derivatives correspond to:
    - intervals of **negative** slope, and
    - intervals where the function is **decreasing**.

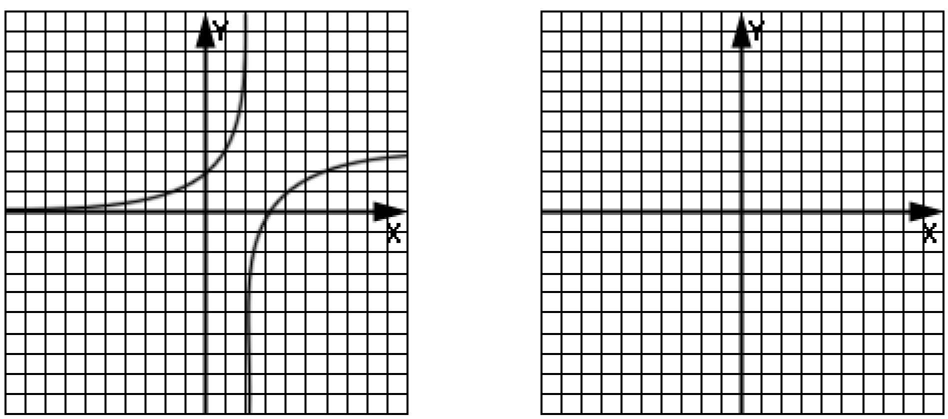
- Intervals of **positive** derivatives correspond to:
    - intervals of **positive slope**, and
    - intervals where the function is **increasing**.

**Assignment:** In problems 1-4, consider the left graph as the original function *f(x)*. On the coordinate system to the right, sketch the graph of *f'(x)*.

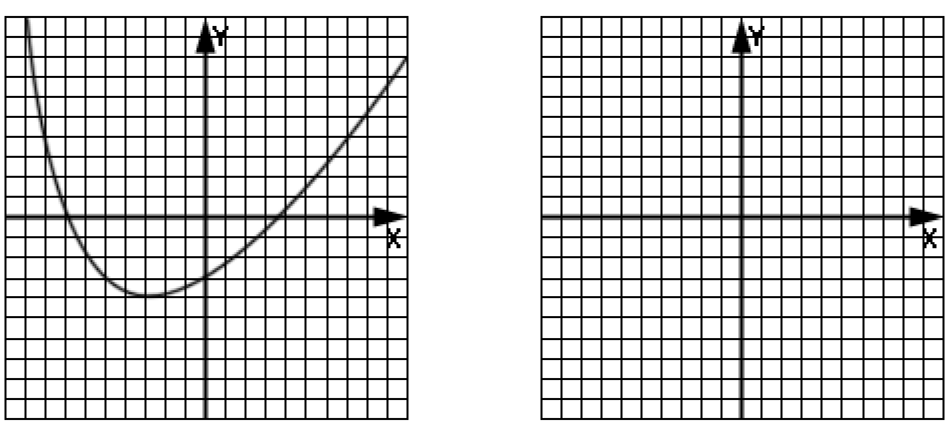
1.



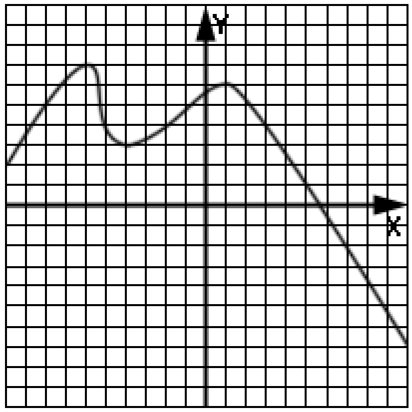

2.

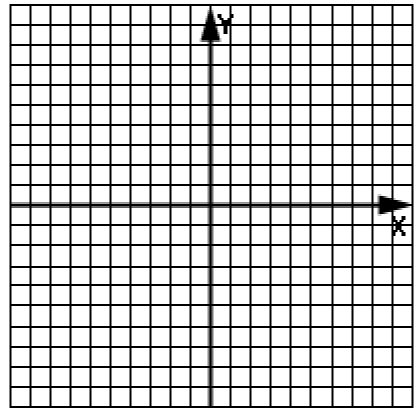



3.



4.

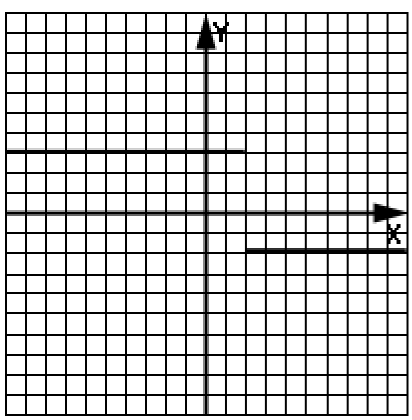


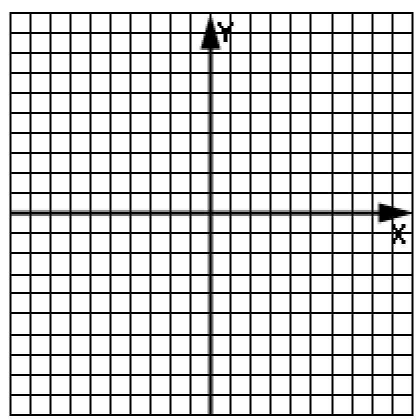




5. Separate the following six items into two associated groups of three items each: Increasing function, Decreasing function, Negative slope, Positive slope, Positive derivative, Negative derivative.

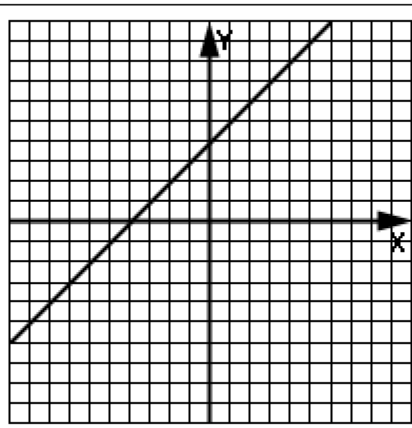
In problems 6-8, given *f′* to the left, sketch the original function *f* to the right.

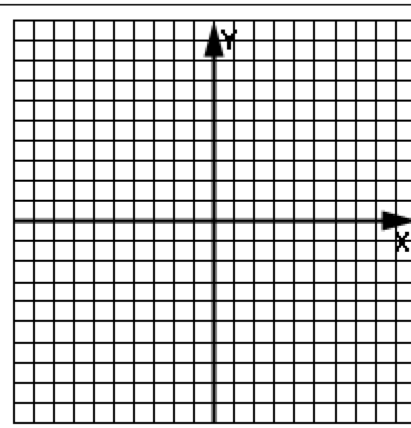
6.

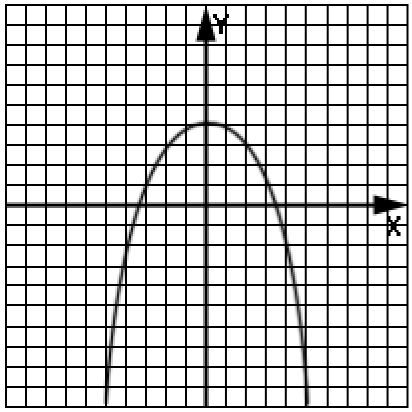


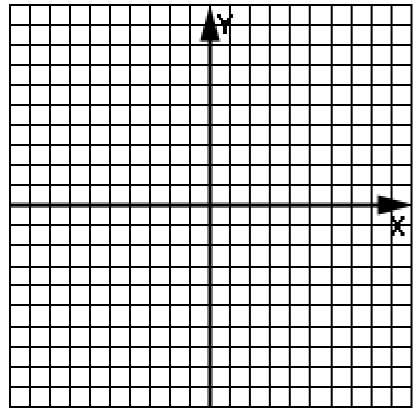




7.

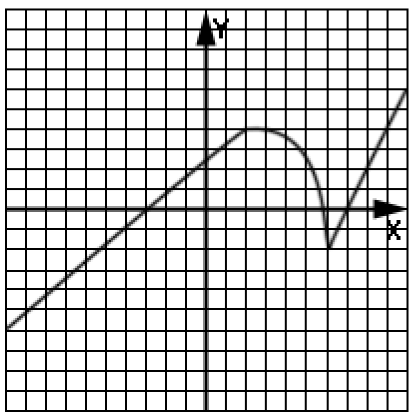


8. 


9. Identify the requested intervals for the function shown here.




a. Interval(s) of negative slope

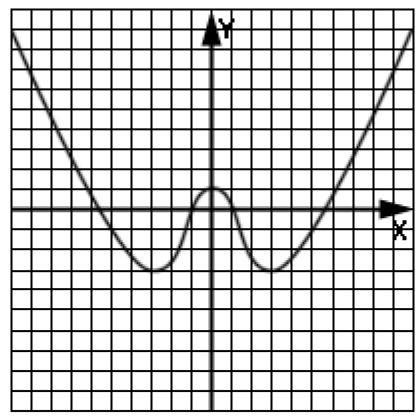
b. Interval(s) of function increase

c. Interval(s) of positive derivative

d. Interval(s) of negative derivative

e. Interval(s) of positive slope

10. Identify the requested intervals for the function shown here.




a. Interval(s) of positive derivative

b. Interval(s) of negative slope

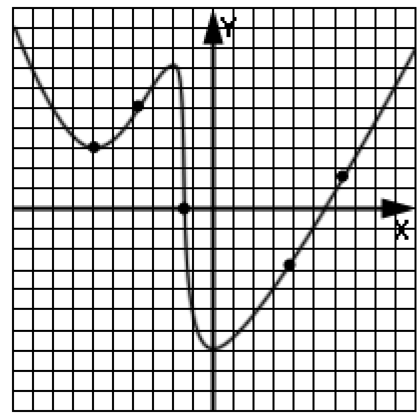
c. Interval(s) of function increase

d. Interval(s) of function decrease

e. Interval(s) of negative derivative

For problems 11 and 12, label the described points directly on the graph.

11.




A. Point A has a positive derivative and a positive function value.

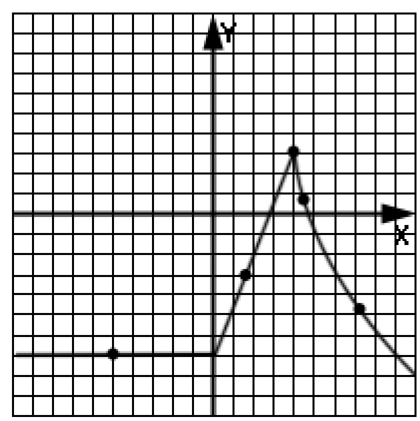
B. Point B has a negative function value and a positive derivative.

C. The slope of the tangent line at point C is 0.

D. Point D has the smallest slope of all the dots.

E. Point E has the largest function value of all the dots.

12.




A. Point A has the largest slope.

B. Point B is on an interval of the function having constant value.

C. Point C has the smallest derivative.

D. Point D has both a negative derivative and a negative function value.

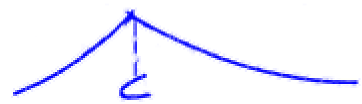
E. The slope for point E cannot be determined.

# Unit 2: Lesson 05 Differentiability

A function is **not differentiable** at x = c if any of the following are true:

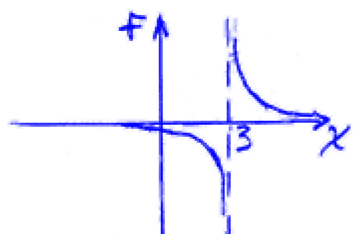
- the function is **discontinuous** at x = c,


- the function has a **cusp** (a sharp turn) at x = c, or



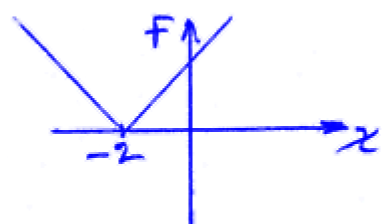

- the function has a vertical tangent line at x = c.


**Example 1:** Sketch the graph of f(x) = 1/(x – 3) and by visual inspection determine any point(s) at which it is not differentiable.




Not diff. at $x = 3$ (Disc)

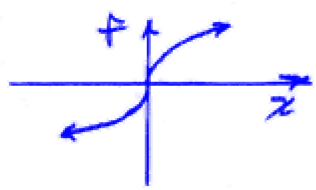
**Example 2:** Sketch the graph of f(x) = |x + 2| and by visual inspection determine any point(s) at which it is not differentiable.




Not diff. at $x = -2$ (cusp)

**Example 3:** Sketch the graph of $f(x) = 4\sqrt[3]{x}$ and by visual inspection determine any point(s) at which it is not differentiable.

$f = 4x^{1/3}$




Not diff at $x = 0$ (vertical tangent line)

**Example 4:** Determine if the function below is differentiable at x = 2.

$$f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ 2x & \text{if } x > 2 \end{cases}$$

$x^2 \to 2^2 = 4$
$2x \to 2 \cdot 2 = 4$ ✓continuos

Left side:

$f'(2) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2}$

$f'(2) = \lim_{x \to 2^-} \frac{x^2 - (2^2)}{x - 2}$

$f'(2) = \lim_{x \to 2^-} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}}$

$= 2 + 2 = 4$

Right side:

$f'(2) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2}$

$= \lim_{x \to 2^+} \frac{2x - (2 \cdot 2)}{x - 2}$

$= \lim_{x \to 2^+} \frac{2\cancel{(x-2)}}{\cancel{x-2}}$

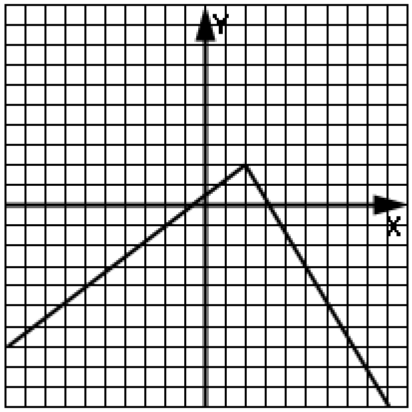
$= 2$

(cusp) Different answers
Limit does not exist

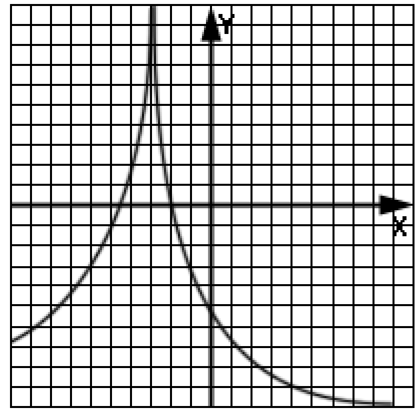
$\boxed{\text{Not diff at } x = 2}$

**Assignment:** In problems 1-6, determine any x-value(s) at which the function is not differentiable and state the reason for non-differentiability.

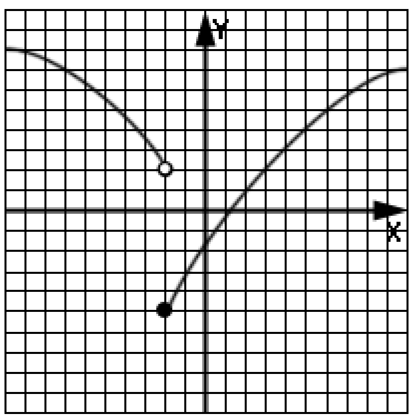
1.



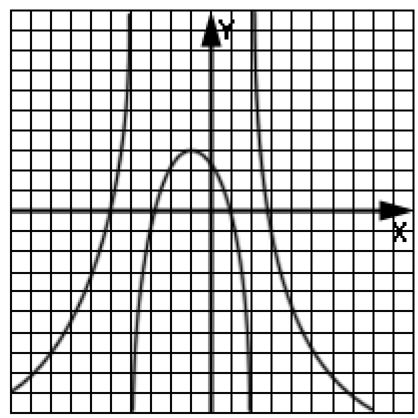

2.



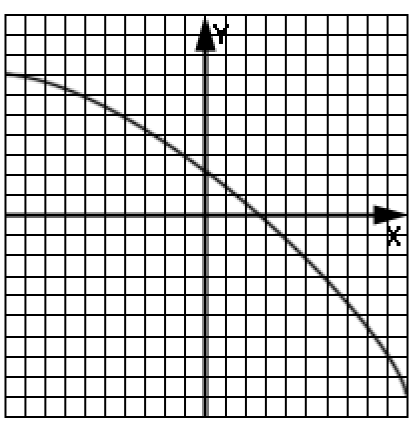

3.



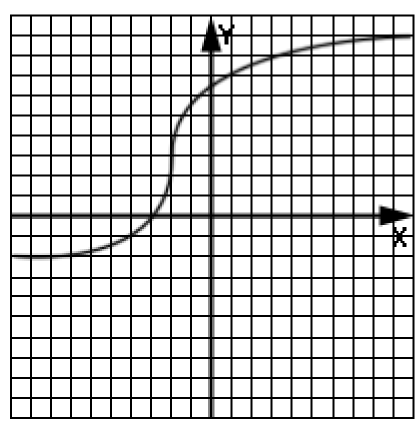

4.




5.




6.



7. Determine if the function below is differentiable at x = 4.

$$f(x) = \begin{cases} x^2 & \text{if } x \le 4 \\ 4x & \text{if } x > 4 \end{cases}$$

---

8. Determine if the function below is differentiable at x = 1.

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x \le 1 \\ x - 1 & \text{if } x > 1 \end{cases}$$

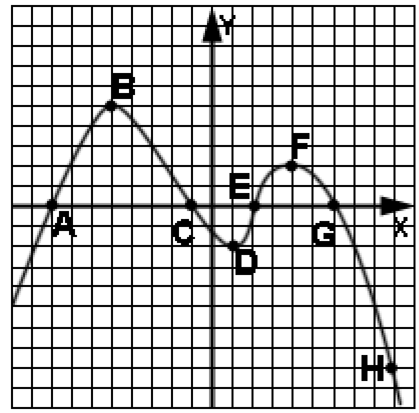
9. Determine if the function below is differentiable at x = 2.

$$f(x) = \begin{Bmatrix} x^2 + 2 & \text{if } x \le 2 \\ 4x - 2 & \text{if } x > 2 \end{Bmatrix}$$

---

10. Determine by analysis if f(x) = |x – 2| is differentiable at x = 2. (Hint: convert to a piecewise function and then compare the left and right derivatives.)

11. Use the labeled points on the graph of this function to answer the questions below.




a. Which point(s) are roots?

b. Which point has the largest derivative?

c. Which point has the smallest derivative?

d. At which point(s) is the slope of the tangent line equal to zero?

e. At which point(s) is there a vertical tangent line?

f. Which point is the largest function value?

g. Which point is the smallest function value?

h. What is the degree of the graphed polynomial?

i. What would be the degree of the derivative of the polynomial whose graph is shown here?

# Unit 2: Review

1. Find the instantaneous rate of change of $f(x) = 7x^2 - 3x$ at $x = 2$.

---

2. Find the average velocity of the time-position function $s(t) = 7t^2 - 3t$ meters between $t = 5$ sec and $t = 8$ sec.

---

3. What is the instantaneous velocity at $t = 5$ sec of the time-position function $s(t) = t^3 - 5$ meters?

4. Find the equation of the tangent line to the curve given by $f(x) = x^2 + 7x - 17$ at $x = -5$.

---

5. What is the equation of the normal line at $x = -5$ of the curve given by $f(x) = x^2 + 7x - 17$?

---

6. Find the equation of the normal line at $x = 1$ of the function $f(x) = \sqrt{x} + 3$.

7. If $y = 3x^2 + 5x - 1$ what is $\frac{dy}{dx}$?

---

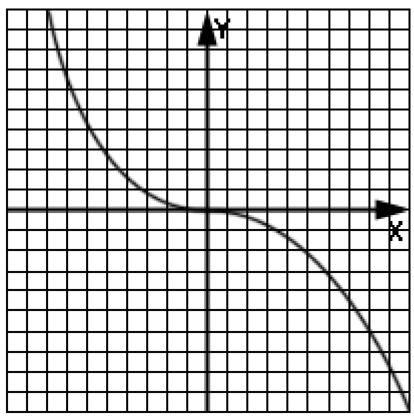
8. Find the derivative of $f(x) = \sqrt{x+2} - 3$ at $x = 7$.

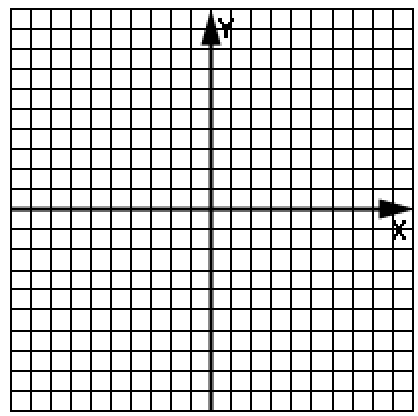
---

9. Use the formal definition of the derivative to find the slope of the normal line to the curve $f(x) = 1/(x + 1)$ at $x = -4$.

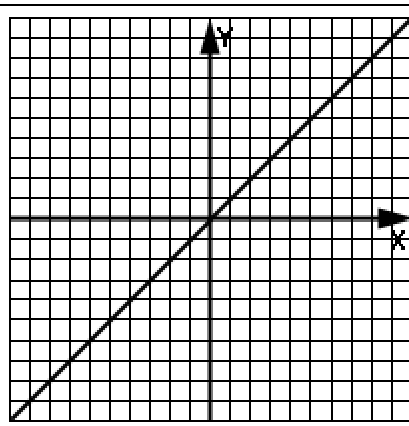
10. Find the function for the velocity where the time-position function is given by $s(t) = t - t^2$ feet. ($t$ is given in minutes).

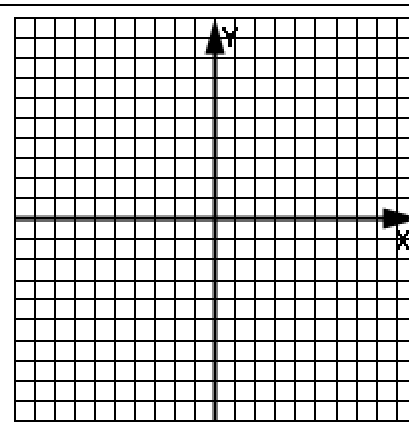
11. The left picture is the function *f*. Sketch its derivative *f'* on the right coordinate system.



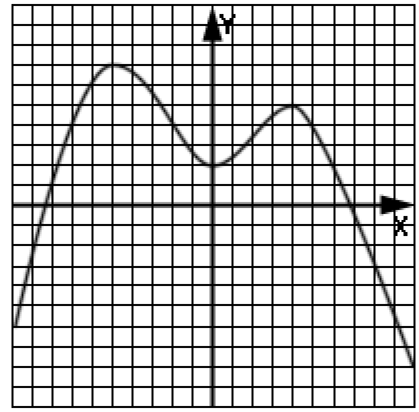

12. The left picture is the function *f'*. Sketch the original function *f* on the right coordinate system.




13. Separate the following six items into two associated groups of three items each: Increasing function, Decreasing function, Negative slope, Positive slope, Positive derivative, Negative derivative.

14. Using the function whose graph is shown to the right, specify the following intervals:



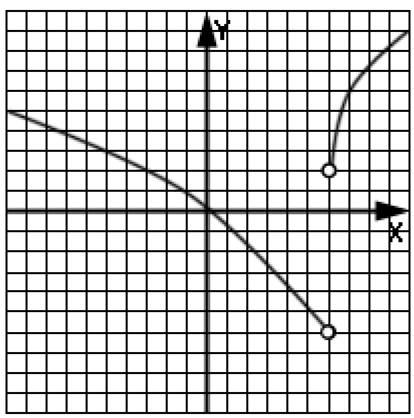

a. Interval(s) of negative derivative

b. Interval(s) of positive slope

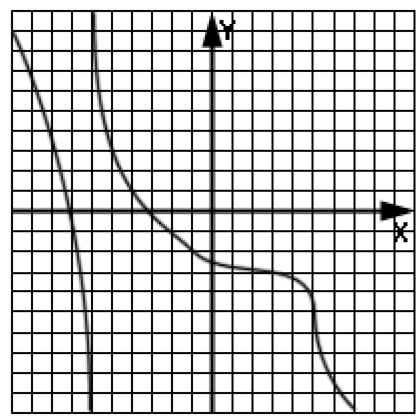
c. Interval(s) of decrease

In problems 15-18, determine the point(s) at which the function is not differentiable. State the reason why.

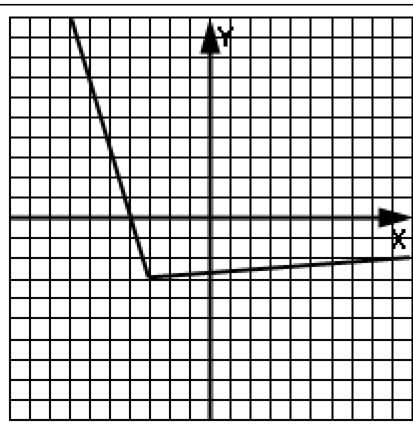
15.



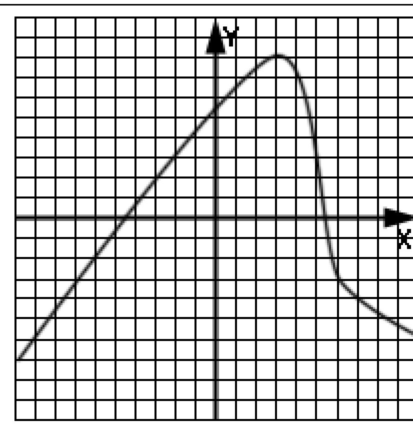

16.




17.




18.

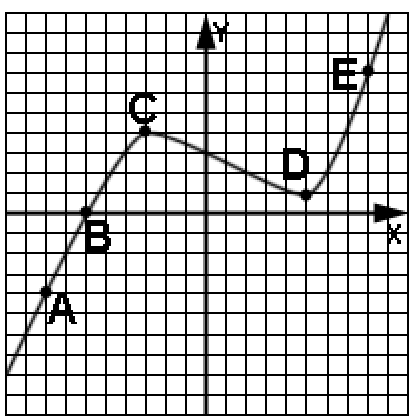


19. Determine by an analysis of a continuity test and "left" & "right" derivatives if this function is differentiable at x = –2.

$$f(x) = \begin{cases} 5x^2 & \text{if } x \le -2 \\ -20x - 20 & \text{if } x > -2 \end{cases}$$

---

20. Use the letters associated with the points on this function to answer the questions.




a. The point(s) at which the tangent line is horizontal.

b. The point(s) at which the function has a root.

c. The point(s) at which the function value is negative and the derivative is positive.

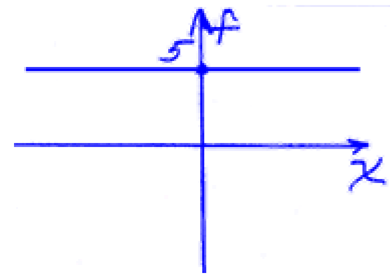
d. The point(s) at which the function value is positive and the slope of the tangent line is positive.

# Calculus, Unit 3

## Derivatives formulas
## Derivative of trig and piecewise functions

# Unit 3: Lesson 01 Constant and power rules

Consider the constant function f(x) = 5.
Clearly the slope is 0 at every point on this "curve", so f'(x) =0.




**Derivative of a constant**:

$f(x) = c$ ;where c is a constant

$\mathbf{f'(x) = 0}$

**Power rule:**

$f(x) = x^n$

$\mathbf{f'(x) = nx^{n-1}}$ ;where *n* can be a positive integer, a negative integer, or fractional

See **Enrichment topic C** for verification of the power rule.

**Miscellaneous rules:**

Because of the rules for limits and since derivatives are fundamentally based on limits, the following rules are easily produced:

If f(x) = cg(x), then **f' = cg'** ;where *c* is a constant
if f(x) = g(x) ± h(x) then **f' = g' ± h'**

In each of examples 1-4, find the derivative of the given function.

**Example 1:** $f(x) = -7x^{1/2} + 22$

$f' = -7\left(\frac{1}{2}\right)x^{-\frac{1}{2}} + 0$

$f'(x) = \boxed{-\frac{7}{2}x^{-1/2}}$

**Example 2:** $f(x) = 3x + 2x^{-5} + 11$

$f'(x) = 3 - 10x^{-6} + 0$

$= \boxed{3 - \frac{10}{x^6}}$

**Example 3:** $y = \sqrt{t^3} - t$

$y = (t^3)^{1/2} - t$

$y = t^{3/2} - t$

$y' = \boxed{\frac{3}{2}t^{\frac{1}{2}} - 1}$

**Example 4:** $g(\alpha) = \dfrac{4}{\alpha^2}$

$g(\alpha) = 4\alpha^{-2}$

$g'(\alpha) = -8\alpha^{-3}$

$= \boxed{\frac{-8}{\alpha^3}}$

**Example 5:** Determine all of the *x* values of the function $f(x) = (1/3)x^3 + x^2 - 35x$ at which tangent lines are horizontal.

$f'(x) = \frac{1}{3}(3)x^2 + 2x - 35 = 0$ slope of horiz tangent line.

$x^2 + 2x - 35 = 0$

$(x+7)(x-5) = 0$

$x + 7 = 0 \qquad x - 5 = 0$

$\boxed{x = -7 \qquad x = 5}$

**Assignment:** In each of problems 1-6, find the derivative of the given function.

1. $f(x) = 18$

2. $f(x) = x^4 - x^2 + 1$

3. $g(x) = \sqrt[3]{x} - 15x$

4. $P(x) = \dfrac{2}{\sqrt[3]{x}}$

5. $h(x) = (4x^4 - x^3 + x)/x$

6. $y = 5t^0 - 7t^3 + t$

7. Find the equation of the tangent line to the curve $y = x^3 - 8x^2 + x - 1$ at $x = 3$.

---

8. What is the equation of the normal line to $f(x) = 11/x$ at $x = -4$?

---

9. Determine all of the *x* values of the function $f(x) = (1/2)x^2 + 5x$ at which tangent lines are horizontal.

---

10. If $s(t) = t^2 - 6t$ feet is the time-position function (with *t* given in seconds), what is the velocity function?

11. What are all of the **numerical** *x* values of the function $f(x) = x^3 - 3x^2$ at which tangent lines have a slope of $\sqrt{2}$ ?

12. What is (are) the *x* position(s) on the curve given by $y = x^2 - 10x + 9$ at which normal lines are exactly vertical?

13. What is the instantaneous rate of change of $f(x)= x^4 - x + 1$ at $x = 2$?

14. Find $h'(-1)$ where $h(t)= t^5 + 6t$.

# Unit 3: Lesson 02 Product and quotient rules

**Product rule:**

If f(x) = u(x) v(x), then

**f'(x) = u · v' + v · u'**

**Example 1:** Find the derivative of $f(x) = \sqrt[3]{x}(x^2 + 5)$.

$f(x) = \sqrt[3]{x}\,(x^2+5) = \underset{u}{x^{1/3}}\,\underset{v}{(x^2+5)}$

$f' = uv' + vu'$

$= \boxed{x^{1/3}(2x) + (x^2+5)\left(\frac{1}{3}x^{-2/3}\right)}$

**Quotient rule:**

If $f(x) = \dfrac{u}{v}$

$$\mathbf{f'(x) = \frac{v \cdot u' - u \cdot v'}{v^2}}$$

See **Enrichment topic D** for verification of both the product and quotient rules.

**Example 2:** Find the derivative of $f(x) = \dfrac{\sqrt{x}}{x + 3x^4}$

$f(x) = \dfrac{\sqrt{x}}{x+3x^4} = \dfrac{x^{1/2}}{x+3x^4}$

$f'(x) = \boxed{\dfrac{(x+3x^4)\left(\frac{1}{2}x^{-1/2}\right) - x^{1/2}(1+12x^3)}{(x+3x^4)^2}} = \dfrac{vu' - uv'}{v^2}$

**Assignment:** In problems 1-8, find the derivatives of the given functions.

1. $f(x) = (5x - 11)/(2x - 1)$

2. $7x(\sqrt{x})$

3. $g(x) = (x + 6)/\sqrt{x}$

4. $L(w) = 7w/(8w^2 + 2)$

5. $h(p) = (p - 1)/(p^2 + 4p - 8)$

6. $f(x) = (-x^3 + 12x^2 - 4)7$

7. $y = (t + 7)(t^7 - 8t^2 + t - 6)$

8. $f(x) = (x - 7)\ 59\ \sqrt[5]{x}$

9. Find the instantaneous rate of change of $f(x) = \sqrt{x}\ \sqrt[5]{x}$ at x = 5.

10. What is the velocity as a function of time (in seconds) of the time-position function $s(t) = (t^3 – 2t)/t^5$ meters?

11. Evaluate $dy/dx$ at $x = 2$ where $y = x/\sqrt{8x}$ .

12. Find the tangent line to $f(x) = (4x – 6)x^{2/3}$ at $x = 8$.

13. Find the y-intercept at x = 1 of the normal line to the curve given by f(x) =

$$\frac{\left(x+\frac{2}{3}\right)}{5\sqrt{x}}.$$

# Unit 3: Lesson 03 — Trig function derivatives

**Trig derivatives:**

$\frac{d}{dx}\sin(x) = \cos(x)$ $\qquad$ $\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$

$\frac{d}{dx}\cos(x) = -\sin(x)$ $\qquad$ $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$

$\frac{d}{dx}\tan(x) = \sec^2(x)$ $\qquad$ $\frac{d}{dx}\cot(x) = -\csc^2(x)$

See **Enrichment topic E** for a derivation of the rules for sine and cosine.

**Example 1:** If f(x) = $x^3\sin(x)$ find f'(x).

$u = x^3 \quad v = \sin(x)$

$f' = uv' + vu'$

$= \boxed{x^3\cos(x) + \sin(x)(3x^2)}$

**Example 2:** If f(x) = sin(x) sec(x) find f'.

$u = \sin(x) \quad v = \sec(x)$

$f' = uv' + vu'$

$f' = \sin(x)\sec(x)\tan(x) + \sec(x)\cos x$

$= \tan^2 x + 1$

$= \boxed{\sec^2 x}$

**Example 3:** Using the identity tan(x) = sin(x)/cos(x) show that the derivative of tan(x) is $\sec^2(x)$.

$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{u}{v} \quad ; \quad f' = \frac{vu' - uv'}{v^2}$

$f' = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)}$

$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \left(\frac{1}{\cos(x)}\right)^2 = \boxed{\sec^2(x)}$

**Assignment:** In problems 1-10, find the derivative of the given function.

1. $f(x) = x \sin(x)$

2. $f(x) = (x^2 + 1)\tan(x)$

3. $f(x) = x^2/\sec(x)$

4. $f(x) = \csc(x)/(x^3 - 8)$

5. $g(t) = \sin(t) \cos(t)$

6. $P(\theta) = 7\tan(\theta)$

7. $f(x) = \dfrac{\cot(x) + 9}{\sqrt{x}}$

8. $f(x) = \dfrac{\sin(x)}{\cos(x) + 2}$

9. $y = t^2(\sin(t) + \cot(t))$

10. $y = \tan(x)\cot(x)$

11. Using an identity for csc(x), show that its derivative is – csc(x) cot(x).

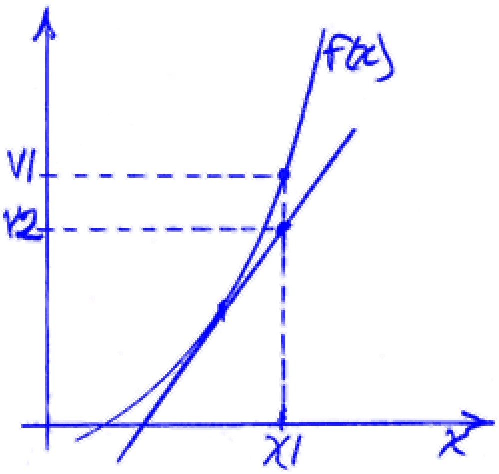
12. Develop the rule for the derivative of cot(x).

# Unit 3: Lesson 04

## Linear approximations
## Derivatives of piecewise functions

The **tangent line** to a curve can be used to obtain an approximation to function values of the curve **near** the point of tangency.

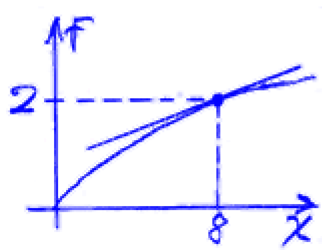
In the adjacent drawing, the true value of the function at *x1* is *v1*. Note that *v2*, the **approximate** value of *f*, is actually the value of the linear function at *x1*.




Notice in the drawing above that the estimate (V2) for *f* is l**ow** because the tangent line is **below** the curve.

Had the tangent line been **above** the curve, the estimate would have been **high**.

**Example 1:** What is a linear approximation to the curve $f(x) = \sqrt[3]{x}$ at x = 8.01? Is this estimate higher or lower than the true value? Why?




Use point $(8,2) = (x_1, y_1)$

$f' = \frac{1}{3}x^{-2/3}$

$f'(8) = \frac{1}{3}(8^{1/3})^{-2} = \frac{1}{3}\cdot 2^{-2} = \frac{1}{12}$

$y - y_1 = m(x - x_1)$

$y - 2 = \frac{1}{12}(x - 8)$

$y = \frac{1}{12}(x-8) + 2$

$y(8.01) = \frac{1}{12}(8.01 - 8) + 2$

$= \boxed{2.00083}$

Estimate is high

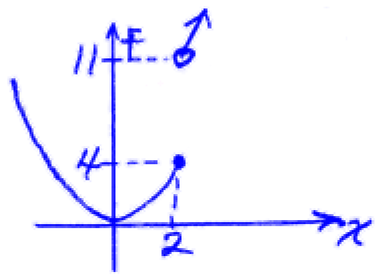
Tangent line is above the curve.

Piecewise functions will naturally result in the derivative also being piecewise.

---

**Example 2:** Find the derivative of the following piecewise function:

$$f(x) = \begin{Bmatrix} x^2 & \text{if } x \le 2 \\ 4x + 3 & \text{if } x > 2 \end{Bmatrix}$$

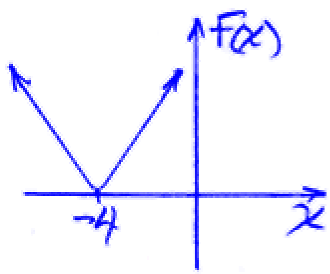



$$f'(x) = \begin{cases} 2x & \text{if } x < 2 \\ 4 & \text{if } x > 2 \end{cases}$$

Absolute value functions are easily represented as piecewise functions.

When asked to take the derivative of an absolute value function, first **convert it to piecewise form**.

**Example 3:** Find the derivative of f(x) = |x + 4|




$$f(x) = \begin{cases} -x-4 & \text{if } x \le -4 \\ x+4 & \text{if } x > -4 \end{cases}$$

$$\boxed{f'(x) = \begin{cases} -1 & \text{if } x < -4 \\ 1 & \text{if } x > -4 \end{cases}}$$

cusp at $x = -4$; Der is not defined there

**Assignment:**

1. Find a linear approximation to the curve $f(x) = -x^{1/2}$ at x = 8.98. Is this estimate higher or lower than the true value? Why?

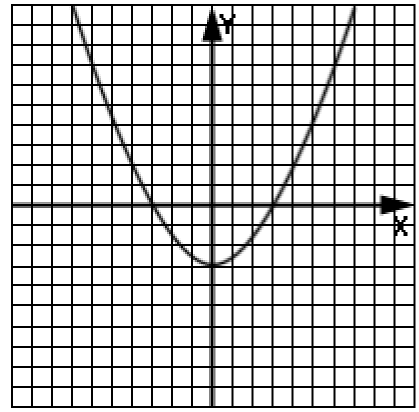
---

2. Find a linear approximation to the curve $f(x) = x^3 - 8x^2 + 12x$ at x = 2.9. Is this estimate higher or lower than the true value? Why?

---

3. Suppose we know that the derivative of a function to be $f'(x) = 2x^2$ and that f(5) = 4. What is a linear approximation for the function value at x = 5.06? Is this estimate higher or lower than the true value? Why?

4. The adjacent graph shows f'(x). Find a linear approximation of f(−3.98) when f(−4) = 5. Is this estimate higher or lower than the true value? Why?




5. Find the derivative of the following piecewise function:

$$f(x) = \begin{cases} \sin(x) & \text{if } x \geq \dfrac{\pi}{2} \\ \tan(x) & \text{if } x < \dfrac{\pi}{2} \end{cases}$$

6. What is the derivative of y = |t|?

7. Find the derivative of a piecewise function that is defined by $f(x) = x^3 + 1$ to the left of $x = -6$ and by $f(x) = 6$ at $x = -6$ and to the right of $x = -6$.

---

8. What is the derivative of $f(x) = |.5x - 3|$?

---

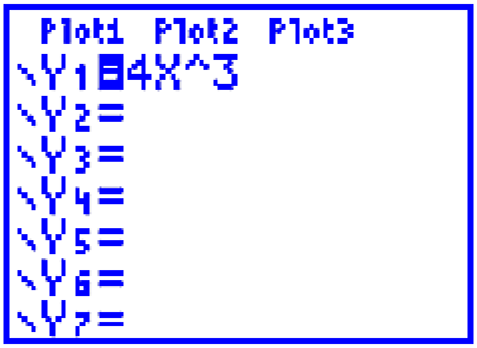
9. If $f(x) = g(x)\,h(x)$ find $f'(5)$ when $g(5) = 3$, $g'(5) = -1$, $h(5) = 22$, and $h'(5) = 4$.

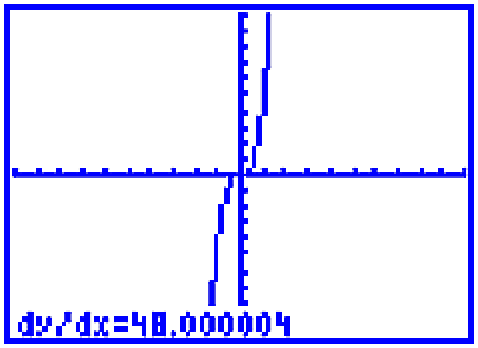
# Unit 3: Lesson 05 — Derivatives on the graphing calculator

See **Calculator Appendix AF** for two techniques for finding the derivative of a function evaluated at a particular point.

The second technique, using **MATH | 8: nDeriv(**, is generally the best and least troublesome.

**Example 1:** Use a calculator to find the derivative of $f(x) = 4x^3$ at x =2. Confirm the calculator answer with a “hand” calculation.




$f'(x) = 12x^2$

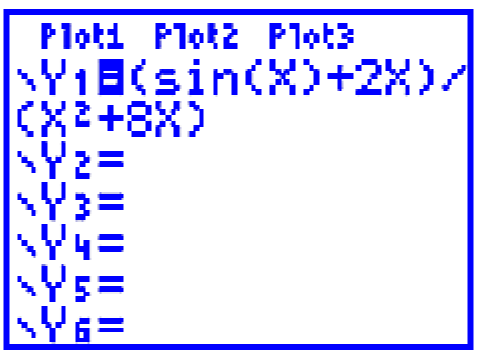
$f'(2) = 12 \cdot 2^2$

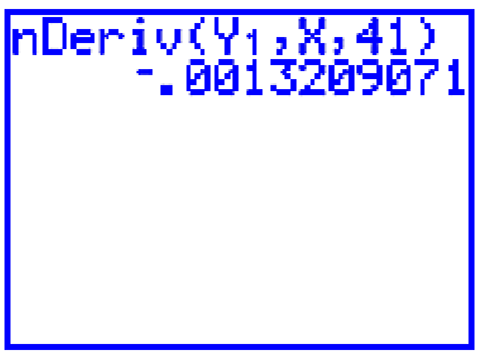
$= 12 \cdot 4$

$= \boxed{48}$

Not exact due to round-off error

**Example 2:** Use a calculator to find the derivative of $f(x) = (\sin(x) + 2x)/(x^2 + 8x)$ evaluated at x = 41. (Assume *x* is in radians.)



**Assignment:**

1. Use a calculator to find the derivative of $f(x) = \sqrt{x}$ evaluated at x = 4. Confirm the calculator answer with a "hand" calculation.

---

2. Use a calculator to find the derivative of $f(x) = x^2$ evaluated at x = 3. Confirm the calculator answer with a "hand" calculation.

---

3. Use a calculator to find the derivative of $f(x) = \sqrt{\tan(x)}\,/(x^2 + 2)$ evaluated at x = –9. (Assume *x* is in radians.)

4. Use a calculator to find the derivative of $\mathrm{y} = \ln(\cos(\mathrm{x}) + \mathrm{x})/\sqrt{\mathrm{x}}$ at x = 22.1. (Assume *x* is in radians.)

# Unit 3:
# Cumulative Review

1. $\lim_{h \to 0} \dfrac{\tan(x+h) - \tan(x)}{h} = ?$ at $x = \pi$ radians.

A. 1

B. 0

C. $\sqrt{3}/2$

D. 1/2

E. None of these

---

2. $\lim_{h \to 0} \dfrac{3(x+h)^2 - 8(x+h) - 5 - 3x^2 + 8x + 5}{h} = ?$

A. $3x^2 - 8x - 5$

B. $6x - 8$

C. 0

D. Undefined

E. None of these

3. $\lim_{h \to 0} \frac{\cos(\pi + h) - \cos(\pi)}{h} = ?$

A. cos(x)

B. sin(x)

C. –sin(x)

D. 0

E. None of these

---

4. State the problem posed by this table in (one-sided) limit notation along with what it seems to be approaching.

| x | f(x) |
|---|---|
| –4.12 | 1/10 |
| –4.11 | –1/100 |
| –4.103 | 1/1000 |
| –4.10054 | –1/100,000 |
| –4.100003 | 1/1,000,000 |

A. $\lim f(x) = \infty$

B. $\lim_{x \to -4^+} f(x) = 0$

C. $\lim_{x \to 4^-} f(x) = \infty$

D. $\lim_{x \to -4.1^+} f(x) = \infty$

E. $\lim_{x \to -4.1^-} f(x) = 0$

F. None of these

5. $\lim_{x \to 16} \frac{16 - x}{\sqrt{x} - 4} = ?$

A. 4 B. -4 C. 0

D. $+\infty$ E. None of these

---

6. $\lim_{x \to 0} \frac{\sin(2x)}{x} = ?$

A. 2x B. x/2 C. 1/2

D. 2 E. None of these

---

7. What is the average rate of change of $f(x) = x^3 – x$ between x = 0 and x = 3?

A. 9 B. $3x^2 – 1$ C. 8

D. ( f(3) – f(0) )/3 E. More than one of these

8. What is the instantaneous rate of change of $f(x) = x^3 - x$ at $x = 3$?

A. $3x^2 - 1$ B. $f'(3)$ C. $f(3)$

D. 26 E. More than one of these

---

9. What is the equation of the normal line to the curve $1/x$ at $x = 2$?

A. $y - 5 = -.25(x - 2)$ B. $y = 4x - 7/2$ C. $y = -.25x + 1$

D. $y = 4x - 15/2$ E. More than one of these

10. What is the velocity of an object whose time-position function is given by s(t) = $(5/4)t^2 - 6t$ meters at t = 6 sec?

A. 9 meters
B. s'(6) m/sec
C. 15 sec
D. 6s'(t) m/sec
E. None of these

# Unit 3: Review

In problems 1-8, find the derivatives of the given functions.

1. $f(x) = 2x^4 - 6x + 11$

2. $f(x) = (x^2 + x)/x$

3. $f(x) = 6\sqrt{x}$

4. $g(t) = t^3(\sqrt{t} + t)$

5. $P(q) = (q^3 + 4q)/(q - \sqrt[3]{q})$

6. $f(\theta) = \sin(\theta)\ (\tan(\theta) + 1)$

7. $f(t) = (t^2+6t)/(t + 1)$

8. $L(x) = (\sin(x) - \csc(x) + x)/(\tan(x) - 4x^3)$

9. Show that the derivative of sec(x) is sec(x) tan(x).

10. What is the derivative of –cos(x) evaluated at π/6 radians?

11. If $f(x) = 12x^2\cot(x)$, find f'(π radians).

12. Find the equation of the tangent line (at x = 1) to the curve given by
$f(x) = (x^2 + 4)\left(\sqrt{x}\right)^3$.

13. Find the equation of the normal line to the curve given by y = 1/x at x = 2.

14. What is a linear approximation to the curve $f(x) = \sqrt{x} + x$ at x = 4.1? Is this estimate higher or lower than the true value? Why?

---

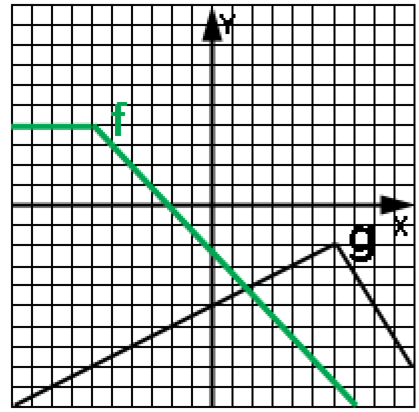
15. What is the derivative of f(x) = |x – 6| + 3?

---

16. Find the derivative of the following piecewise function:

$$f(x) = \begin{cases} -x^3 + 2x & \text{if } x < -3 \\ x^2 & \text{if } x \geq -3 \end{cases}$$

17. Using the functions f(x) and g(x) from the adjacent graph, find p'(–4) where p(x) = f(x) g(x).



# Calculus, Unit 4

## Chain rule
## Higher order derivatives
## Applied rates of change

# Unit 4: Lesson 01 Chain rule fundamentals

**Chain rule:**

If F(x) = f( g(x) ) and f'(x) and g'(x) both exist, then

**F'(x) = f'( g(x) ) · g'(x)**

See **Enrichment Topic F** for a justification of this rule.

**In the following examples, find the derivative of the given functions:**

**Example 1:** f(x) = $(3x + 9)^4$

$$f'(x) = \underbrace{4(3x+9)^3}_{f'(g)}\ \underbrace{3}_{g'}$$

$$= \boxed{12(3x+9)^3}$$

---

**Example 2:** f(x) = $\dfrac{2}{\sqrt[3]{4x-7}}$

$$f(x) = 2(4x-7)^{-1/3}$$

$$f'(x) = \underbrace{2\left(-\tfrac{1}{3}\right)(4x-7)^{-4/3}}_{f'(g)}\ \underbrace{(4)}_{g'}$$

$$= -\frac{8}{3}(4x-7)^{-4/3} = \frac{-8}{3(4x-7)^{4/3}}\ \frac{(4x-7)^{2/3}}{(4x-7)^{2/3}}$$

$$= \boxed{\frac{-8(4x-7)^{2/3}}{3(4x-7)^2}}$$

**Example 3:** f(x) = $\frac{2x}{(5x+1)^3}$

$$f(x) = \underbrace{2x}_{u}\underbrace{(5x+1)^{-3}}_{v}$$

$$f'(x) = \underbrace{2x}_{u}\underbrace{(-3)(5x+1)^{-4}(5)}_{v'} + \underbrace{(5x+1)^{-3}}_{v}\underbrace{2}_{u'}$$

$$f'(x) = \boxed{-30x(5x+1)^{-4} + 2(5x+1)^{-3}}$$

---

**Example 4:** f(x) = $8x^5\,\sqrt{x^2-6}$

$$f(x) = 8x^5(x^2-6)^{1/2}$$

$$f' = \underbrace{8x^5}_{u}\underbrace{\tfrac{1}{2}(x^2-6)^{-1/2}(2x)}_{v'} + \underbrace{(x^2-6)^{1/2}}_{v}\underbrace{8\cdot 5x^4}_{u'}$$

$$f' = 8x^6(x^2-6)^{-1/2} + 40x^4(x^2-6)^{1/2}$$

multiply by $(x^2-6)^{1/2}/(x^2-6)^{1/2}$

$$f' = \boxed{\frac{8x^6(x^2-6)^{1/2}}{x^2-6} + 40x^4(x^2-6)^{1/2}}$$

**Assignment:** In problems 1-12, find the derivative of the given function:

1. $f(x) = 4(3x - 7)^2$

2. $g(x) = (2 - 6x)^5$

3. $h(x) = (2 - 6x^2)^5$

4. $f(x) = \sqrt{2x + 4}$

5. $g(x) = 8x(1+8x)^{1/3}$

6. $m(x) = (3x^3 - 6x)(x^2 + 1)$

7. f(x) = $9x\sqrt{2+x^2}$

8. f(x) = 1/(2x – 1)

9. f(x) = $\left(\frac{-1}{3x+5}\right)^3$

10. f(x) =11/(x – 22)

11. f(x) = (25x$^{-3}$+1)/(2x – 1)

12. h(x) = $\frac{\sqrt{x^2+1}}{4x}$

13. Evaluate f ’(x) = at $x = 1$ where $f(x) = x(x^3 - x + 5)^{3/2}$.

14. If $f(x) = x/(x + 1)$ find f ‘(3).

**Unit 4: Lesson 02**

# Chain rule applied to trig functions

**Example 1:** Find the derivative of f(x) = $\sin^2(x)\tan(x)$

$$f(x) = \underbrace{(\sin(x))^2}_{u}\underbrace{\tan(x)}_{v}$$

$$f'(x) = \sin^2(x)\sec^2 x + \tan(x)\,2(\sin(x))^1\cos(x)$$

$$f'(x) = \tan^2 x + 2\frac{\sin(x)}{\cancel{\cos(x)}}\sin(x)\cancel{\cos(x)}$$

$$f'(x) = \boxed{\tan^2 x + 2\sin^2 x}$$

**Example 2:** Find the derivative of f(x) = $\left(\sqrt{\sec(x)} - 1\right)(x^2+x)^6$

$$f(x) = \underbrace{((\sec(x))^{1/2} - 1)}_{u}\underbrace{(x^2+x)^6}_{v}$$

$$\boxed{f'(x) = \underbrace{((\sec(x))^{1/2}-1)}_{u}\underbrace{6(x^2+x)^5(2x+1)}_{v'} + \underbrace{(x^2+x)^6}_{v}\underbrace{\left(\tfrac{1}{2}(\sec(x))^{-\frac{1}{2}}\sec x\tan x\right)}_{u'}}$$

**Example 3:** Find the equation of the tangent line to the curve given f(x) = $x\sqrt{\sin(x)}$ at x = 2 radians.

$$f(x) = x\,(\sin(x))^{1/2}$$

$$f'(x) = \underbrace{x}_{u}\,\underbrace{\tfrac{1}{2}(\sin(x))^{-1/2}(\cos(x))}_{v'} + \underbrace{(\sin(x))^{1/2}}_{v}\,\underbrace{1}_{u'}$$

$$f'(2) = .51716 = m$$

$$f(2) = 2\sqrt{\sin(2)}$$
$$= 1.907$$
$$(x_1, y_1) = (2, 1.907)$$

$$y - y_1 = m(x - x_1)$$

$$\boxed{y - 1.907 = .51716(x - 2)}$$

**Assignment:** In the following problems, find the derivative of the given function:

1. $g(x) = \cos^2(x)$

2. $m(x) = \cot(4x^2 - 1)$

3. $f(x) = \tan^2(x)\sin(2x + 1)$

4. $f(x) = \sin(x)/(x - \cos(3x))$

5. $f(x) = \cos(\tan(x))\sec(x)$

---

6. $h(x) = 11x^2\cos(5x)$

---

7. $f(x) = \sqrt{\sin(x) + 1}$

8. Evaluate the derivative of $f(x) = \cos(3x)$ at $\pi/6$ radians.

9. Find the equation of the tangent line to $f(x) = 3\sin(x)/x$ at $x = \pi/2$ radians.

10. What is the equation of the normal line to $f(x) = -\cos(4x)$ at $x = \pi/4$ radians?

11. If $f'(g(3)) = 4$, $g'(f(3)) = -5$, $f'(3) = 7$, $g'(3) = 10$, $f(3) = -11$, $g(3) = 2$, find $G'(x)$ evaluated at $x = 3$ where $G(x) = g(f(x))$.

**Unit 4: Lesson 03** **Higher order derivatives**

Notation for higher order derivatives:

The **first derivative** of y = f(x) is denoted as **y'**, $\frac{dy}{dx}$, $y^{(1)}$, or **f '(x)**.

The **second derivative** of *y* (the derivative of y') is denoted as **y''**, $\frac{d^2y}{dx^2}$, $y^{(2)}$, or **f ''(x)**.

The **third derivative** of *y* (the derivative of y'') is denoted as **y'''**, $\frac{d^3y}{dx^3}$, or $y^{(3)}$, or **f '''(x)**.

etc.

Notice that for a polynomial as we progress from the original function to higher order derivatives, each derivative produces a polynomial that is one degree less than the previous one.

---

**Example 1:** Find the first three derivatives of $f(x) = 5x^4 - 6x^3 - x^2 + x + 17$

$f' = 20x^3 - 18x^2 - 2x + 1$

$f'' = 60x^2 - 36x - 2$

$f''' = 120x - 36$

Notice degree is one less each time

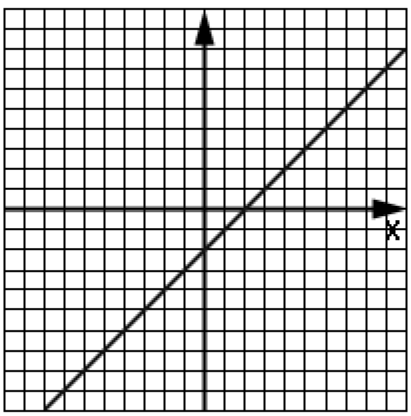
**Example 2:** If y = sin(6x) – 1, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \boxed{\cos(6x)6}$$

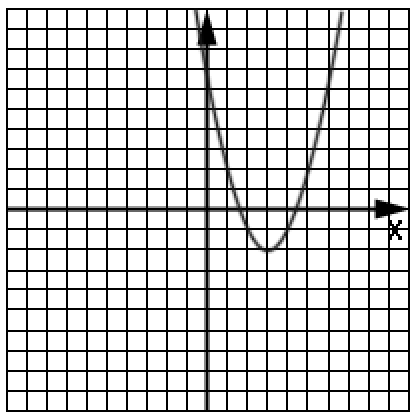
$$\frac{d^2y}{dx^2} = -6\sin(6x)\,6 = \boxed{-36\sin(6x)}$$

---

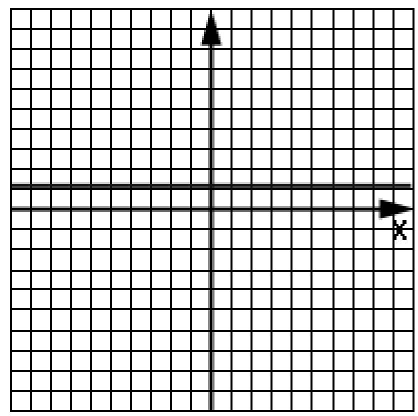
**Example 3:** Of the three graphs below, one is the original function f(x), another is f ’(x), and the third is f ’’(x). Identify which is which.




f'(x)




f(x)




f''(x)

---

**Example 4:** Write a formula for the nth derivative of y = 2/(3x – 7).

$$y = 2(3x-7)^{-1}$$

$$y' = 2(-1)(3x-7)^{-2}(3)$$

$$y'' = 2(-1)(-2)(3x-7)^{-3}3^2$$

$$y''' = 2\underbrace{(-1)(-2)(-3)}_{(-1)^n n!}\underbrace{(3x-7)^{-4}}_{(3x-7)^{-(n+1)}}\underbrace{3^3}_{3^n}$$

$$\boxed{\frac{d^ny}{dx^n} = 2(-1)^n n!(3x-7)^{-(n+1)}3^n}$$

**Assignment:**

1. Find the second derivative of $y = 4x^3$.

2. If $f(x) = x\cos(x)$, find $f^{(1)}$, $f^{(2)}$, and $f^{(3)}$.

3. Find $\frac{d^{301}f}{dx^{301}}$ for $f(x) = e^x$. (Use the fact that $f'(x) = e^x$.)

4. Find $y'$ and $y''$ for $y(x) = \tan(x)$.

5. What is the 2$^{nd}$ derivative of $x^2(x + 5)^{1/2}$ ?

6. Realizing that the velocity is the first derivative of the position function and that acceleration is the 2nd derivative, find the velocity and acceleration of an object at t = 3 sec if its position is given by $s(t) = 4t^3 - 6t$ meters.

---

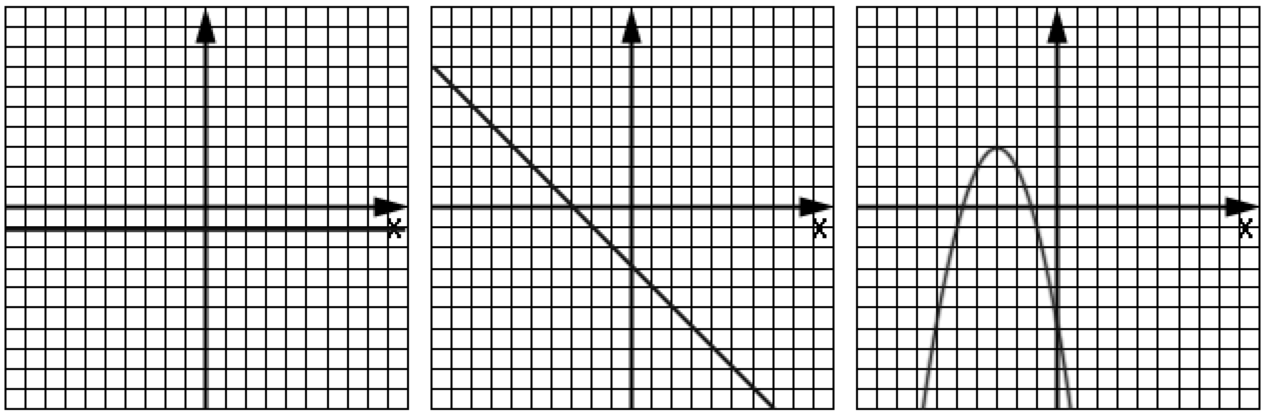
7. Evaluate $\frac{d^2g}{dx^2}$ at $x = \pi$ radians when $g(x) = \sec(x)$.

---

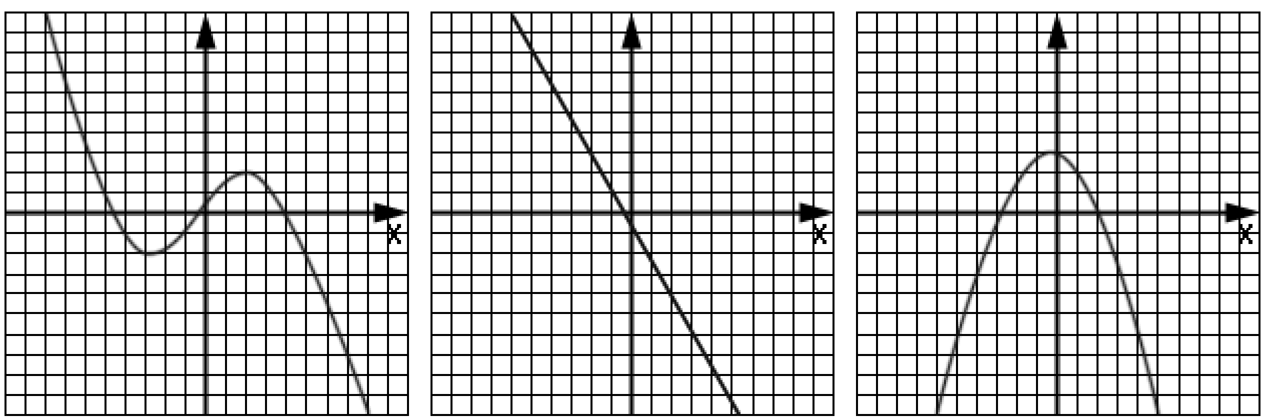
8. Write a formula for the $n^{th}$ derivative of $f(x) = 1/(x + 5)$.

In problems 9 and 10, three graphs are given in random order. Identify each as either the original function f(x), f ’(x), or f ’’(x).

9.




10.




11. Find $f^{(3)}$ where $f(x) = (5x + 1)^{-1}$.

**Unit 4: Lesson 04**

# Applied rates of change
# Velocity, speed, and acceleration

**Rectilinear motion** (motion of an object along a straight line):

- **Position** is the location of an object and is given as a function of time. Conventional notation uses s(t).

  **Displacement** is the difference between the final position and the initial position… displ = $s(t_f) - s(t_i)$.

  **Total distance traveled**… Sum of each distance between turns.

- **Velocity** is the rate of change of position… v(t) = s'(t).

  **Average speed** over the interval $[t_i, t]$ is Δs/Δt.

  **Speed** is the absolute value of velocity… |v(t)|

- **Acceleration** is the rate of change of velocity … a(t) = v'(t) = s''(t).

In the following examples, assume rectilinear motion as described by $s(t) = 4t^3 - 40t^2 + 50t + 2$ meters over the time interval [0 sec, 10 sec].

---

**Example 1:** Find velocity, speed, and acceleration as functions of time. Label each with the correct units.

$V(t) = s'(t) = (12t^2 - 80t + 50)$ m/sec

$speed(t) = |12t^2 - 80t + 50|$ m/sec

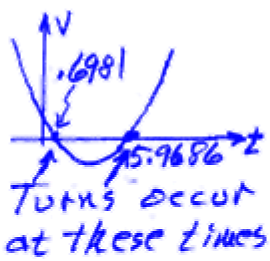
$a(t) = v'(t) = s''(t) = (24t - 80)$ m/sec$^2$

**Example 2:** Find the displacement of the object over the interval [0 sec, 10sec].

$$\begin{aligned} disp &= S(10) - S(0) \\ &= 4\cdot 10^3 - 40(10)^2 + 50\cdot 10 + 2 - (4\cdot 0^3 - 40\cdot 0^2 + 50\cdot 0 + 2) \\ &= 4{,}000 - 4000 + 500 + 2 - 2 \\ &= \boxed{500\,m} \end{aligned}$$

---

**Example 3:** What is the total distance traveled?




$$\begin{aligned} D_1 &= |S(.6981) - S(0)| = 16.772 \\ D_2 &= |S(5.9686) - S(.6981)| = 292.803 \\ D_3 &= |S(10) - S(5.9686)| = 776.031 \\ Dist_{Tot} &= D_1 + D_2 + D_3 \\ &= \boxed{1085.606\,m} \end{aligned}$$

---

**Example 4:** Describe the motion of the object in terms of advancing and/or retreating.

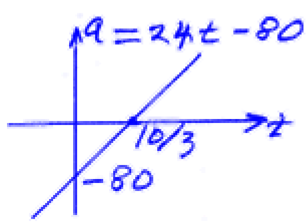
Advancing $(v>0) \to [0, .6981)$ sec, $(5.9686, 10]$ sec

Retreating $(v<0) \to (.6981, 5.9686)$ sec

see drawing in Ex 3

---

**Example 5:** Describe when the object is accelerating and/or decelerating.




accelerating $(a>0) \to (10/3, 10]$ sec

decelerating $(a<0) \to [0, 10/3)$ sec

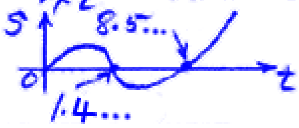
**Example 6:** For what values of *t* is the average velocity equal to zero?

$$V_{AV} = \frac{s(t) - s(0)}{t - 0} = 0$$

Solve $s(t) - s(0) = 0$

$$4t^3 - 40t^2 + 50t + \cancel{2} - \cancel{2} = 0$$

$$4t^3 - 40t^2 + 50t = 0$$




$$t = \boxed{\begin{array}{l}1.46447\text{ sec} \\ \text{and } 8.53553\text{ sec}\end{array}}$$

**Speeding up**… when velocity and acceleration have the same signs.

**Slowing down**… when velocity and acceleration have opposite signs.

**Advancing**… when velocity is positive.

**Retreating**… when velocity is negative.

**Accelerating**… when acceleration is positive.

**Decelerating**… when acceleration is negative.

**Assignment:**

For problems 1 – 6, assume that an object moves rectilinearly according to the position function s(t) = 4sin(3t) – 3t + 1 meters over the time interval [–1 sec, 1 sec].

1. Find velocity, speed, and acceleration as functions of time and label with the appropriate units.

2. Find the displacement over the interval [–1 sec, 1 sec].

3. What is the total distance traveled?

4. Describe the motion of the object in terms of advancing and/or retreating.

5. Describe where the object is accelerating and/or decelerating.

6. For what values of *t* is the average velocity equal to zero?

For problems 7 – 10, assume that an object is falling vertically under the influence of gravity according to the position function $s(t) = 100 - 10t - 16t^2$ feet over the time interval [0 sec, 2 sec].

7. Find velocity, speed, and acceleration as functions of time and label with the appropriate units.

8. What is the position, velocity, speed, and acceleration at t = 2 seconds?

9. What is the total distance traveled over the interval?

10. What is the average velocity over the interval?

# Unit 4: Cumulative Review

1. Find the point(s) on the graph of the function f(x) = $x^3 - 4$ where the slope is 3.

   A. (1, 3)  B. (1,3), (–1, –3)  C. (1, –3), (–1, –5)  D. $(0, \sqrt[3]{3}, -1)$

---

2. If f(x) = ( 1 + cos(x) )/( 1– cos(x) ) what is f ′(x)?

   A. –2/sin(x)  B. $-2\sin(x)/(1-\cos(x))^2$  C. 2/sin(x)  D. sec(x)

---

3. If f(x) = $x^2g(x)$ which expression is the correct representation of f′(x)?

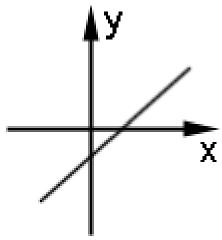
   A. ( x g′(x) + 2g(x) )x  B. ( x g(x) + 2g′(x) )x  C. 2xg′(x)  D. $x^2g'(x)$

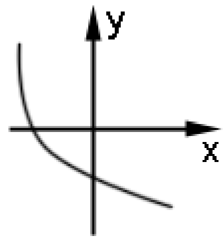
---

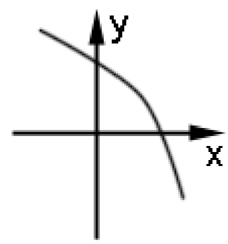
4. The local linear approximation of a function will always be less than or equal to the function's value if, for all *x* in an interval containing the point of tangency

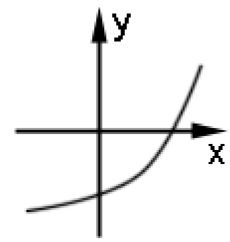
   A. the curve is concave upward.
   B. the curve is concave downward.
   C. the tangent line is exactly horizontal.
   D. the tangent line is exactly vertical.

5. Which of the functions below is decreasing while its rate of change is also decreasing?

A.



B.



C.



D.
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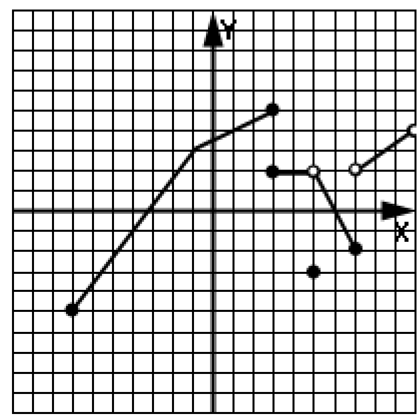
6. In the graph below, which of the following statements are true? (It is possible to have more than one answer.)




A. $\lim_{x\to 7}(y)$ exists
B. The relation is double-valued at x = 3.
C. $\lim_{x\to 5}(y)$ exists
D. y is not defined at x = 5.
E. y(7) = –2
F. The domain of this relation is [–7, 10].
G. The derivative is never negative.
H. The derivative is defined everywhere in the interval (–7, 3).
I. The derivative is defined everywhere in the interval [–7, 3].

---

7. $\lim_{x\to 0}(\sin(5x)/x)$ = ?

A. 0   B. Does not exist   C. 1   D. 5

---

8. $\lim\limits_{x\to 4}\dfrac{f(4)-f(x)}{4-x} = 7.002$ indicates that at x = 4

A. f(x) is increasing.   B. f(x) is decreasing.   C. f(x) is constant.

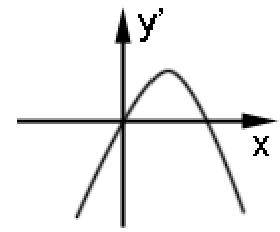
D. f(x) is undefined.

9. Use $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ to find f ’(2) where f(x) = $3x^2 - 4$.

---

10. Use $f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ to find f ’(x) where f(x) = $5x^2 - x$.

11. The adjacent graph represents the derivative of which one of the these functions?




A. $x^3 + x^2$
B. $x^3 - x^2$
C. $-x^3 + x^2$
D. $-x^3 - x^2$

---

12. For what value(s) of x is the slope of the tangent line to $y = x^2 + 2x - 6$ equal to $-4$?

**Unit 4: Review**

In each of problems 1-6, find the first derivative of the given function.

1. $f(x) = (5x^2 + x)^3$

2. $g(x) = 8x^4\sqrt{x^2 - 7}$

3. $q(x) = (8x - 1)^{-1}$

4. $p(x) = (\,x \sin(x)\,)^2$

5. Evaluate $f'(-3)$ where $f(x) = (3x - 2)^2$.

6. $f(x) = \sin(\,\cos(x)\,)$

7. Assume $f(0) = 2$, $f'(0) = -4$, $f''(0) = 3$, and $g(x) = \tan(4x)\, f(x)$. What is the equation of the line tangent to $g(x)$ at $x = 0$?

---

8. Assume $f(3) = 1$, $f'(3) = 4$, $f(2) = -5$, $f'(2) = -9$, $g(2) = 3$, $g'(2) = -11$, $g(3) = 1$, and $g'(3) = 0$. Find the value of the derivative of the composite function $m(x) = f(\,g(x)\,)$ at $x = 2$.

---

9. Find the equation of the normal line to $f(x) = \sqrt{\tan(x)}$ at $x = \pi/3$ radians.

10. $f(x) = \sin^2(3x)$; $f'(x) = ?$

11. Find y' when $y = \cos(2x)\csc(3x)$

12. Find $\frac{d^3y}{dx^3}$ when $y = 5\cos(4x)$.

13. Find $\frac{d^2y}{dx^2}$ when $y = \sqrt{x}\,\sin(\sqrt{x})$ evaluated at 4 radians.

In problems 14-19, assume an object is moving rectilinearly in time according to $s(t) = 4t^2 - 6t + 1$ meters over the time interval [0, 4] seconds.

14. Find the velocity, speed, and acceleration as functions of time and give the appropriate units of each.

| | |
|---|---|
| 15. What is the velocity at t = 1 sec? | 16. What is the acceleration at t = 1 sec? |

17. On what time interval(s) is the particle advancing (moving to the right) and retreating? Justify your answers.

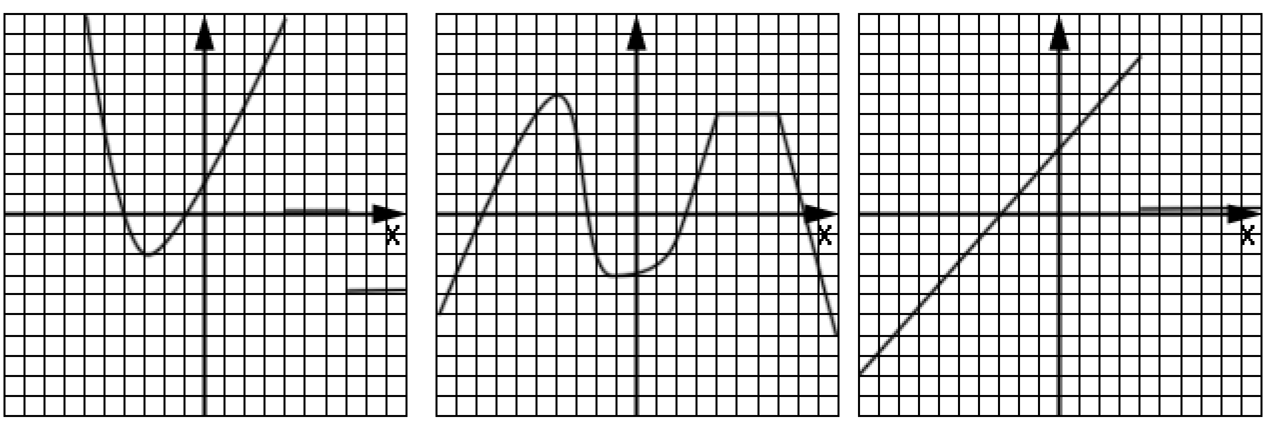
18. What is the total distance traveled?

19. Use the data in this chart to find g'(1) where g(x) = f(2x).

| x | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| f '(x) | .2 | .3 | .5 | .7 | .9 | 1 |

20. Write a formula for the $n^{th}$ derivative of f(x) = $\sqrt{2x-7}$.

21. The three graphs below are arranged in random order and represent a function, f(x), and its first two derivatives ( f '(x) and f ''(x) ). Identify each graph directly by labeling f, f ' or f '' directly below the graph.



# Calculus, Unit 5

# Implicit Differentiation

**Unit 5: Lesson 01** **Implicit differentiation fundamentals**

Consider finding y' where y is given by the equation xy + cos(x) y = –3.

Traditional solution (**solve for y** and find the derivative w.r.t. *x*):

$$y(x+\cos(x))=-3$$
$$y=-3(x+\cos(x))^{-1}$$
$$y'=\boxed{3(x+\cos(x))^{-2}(1-\sin(x))}$$

Implicit derivative: Take the derivative w.r.t. *x* of each term while **treating *y* as some unknown function of x** (i.e. *y* is an **implicit** function of *x*).

$$xy'+y(1)+\cos(x)y'+y(-\sin(x))=0$$
$$xy'+\cos(x)y'=y\sin(x)-y$$
$$y'(x+\cos(x))=y\sin(x)-y$$
$$y'=\boxed{\frac{y\sin(x)-y}{x+\cos(x)}}$$

The traditional solution above works well if it is easy to solve for *y*. Consider example 1 below in which it is not easy to solve for *y*.

**Example 1:** Solve $x^2 + 4y^3 = x + 2y$ implicitly for y'.

$$2x+12y^2y'=1+2y'$$
$$12y^2y'-2y'=1-2x$$
$$y'(12y^2-2)=1-2x$$
$$y'=\boxed{\frac{1-2x}{12y^2-2}}$$

From example 1 we see that the following steps are appropriate when **implicitly finding the derivative** of *y* with respect to (w.r.t.) *x*.

- Take the derivative of each term with respect to *x*.
- Move all y' terms to one side and all other terms to the other side.
- Factor out y'.
- Solve for y'.

**Example 2:** Solve $x^3 + xy = y^2 + 1$ implicitly for y'.

$$3x^2 + xy' + y(1) = 2yy' + 0$$
$$xy' - 2yy' = -3x^2 - y$$
$$y'(x - 2y) = -3x^2 - y$$
$$y' = \boxed{\frac{y + 3x^2}{2y - x}}$$

---

**Example 3:** Solve $\sin(xy^2) + x = 5y$ implicitly for y'.

$$\cos(xy^2)[x2yy' + y^2 1] + 1 = 5y'$$
$$2xyy'\cos(xy^2) + y^2\cos(xy^2) + 1 = 5y'$$
$$2xyy'\cos(xy^2) - 5y' = -y^2\cos(xy^2) - 1$$
$$y'(2xy\cos(xy^2) - 5) = -y^2\cos(xy^2) - 1$$
$$y' = \frac{-y^2\cos(xy^2) - 1}{2xy\cos(xy^2) - 5} = \boxed{\frac{y^2\cos(xy^2) + 1}{5 - 2xy\cos(xy^2)}}$$

**Example 4:** Find y' implicitly from $y^{2/3} + x^{2/3} = 10$ and then evaluate the derivative at (1, 27).

$$\frac{2}{3}y^{-1/3}y' + \frac{2}{3}x^{-1/3} = 0$$

$$\cancel{\frac{2}{3}}y^{-1/3}y' = -\frac{\cancel{2}}{\cancel{3}x^{1/3}}$$

$$y' = \frac{-y^{1/3}}{x^{1/3}}$$

sub in (1, 27)

$$y' = \frac{-27^{1/3}}{1^{1/3}} = \frac{-3}{1} = \boxed{-3}$$

**Assignment:** In each problem, implicitly find dy/dx.

1. $x^3 + y^2 = 4$

2. $x^{1/3} + y^{1/2} = 10$

3. $\cos(xy) = y$

4. $3x + 1 = y^2$

5. $\sin(y) + x(1 + \cot(x)) = -4$

6. $\cos(x) = 2 - 11\cos(3y)$

7. $\tan(x^2 + y) = (2x - 3)y^3$

8. $(x + 7y)/y^2 = x^4 - xy$

9. Implicitly find y' from $x^2 + y^2 - xy = 21$ and then evaluate the derivative at (4, –1).

10. Implicitly find y' from $\sin(-x + y) = x$ and then evaluate the derivative at the origin.

# Unit 5: Lesson 02 — Tangent and normal lines (with implicit derivatives)
## Implicit higher order derivatives

**Example 1:** Find the equation of the tangent line to the curve given by $y^3 - 1 = 4xy - x^3$ at (0, 1).

$$3y^2y' = 4xy' + y4 - 3x^2$$
$$3y^2y' - 4xy' = 4y - 3x^2$$
$$y'(3y^2 - 4x) = 4y - 3x^2$$
$$y' = (4y - 3x^2)/(3y^2 - 4x)$$

sub in (0,1)

$$m = y' = (4\cdot 1 - 3\cdot 0^2)/(3\cdot 1^2 - 4\cdot 0)$$
$$= 4/3$$

$(x_1, y_1) = (0,1)$

$$y - y_1 = m(x - x_1)$$
$$\boxed{y - 1 = \frac{4}{3}(x)}$$

---

**Example 2:** At what *x* value(s) is the tangent line horizontal on the curve given by $4x^2 + y^2 - 12x = -4y - 4$?

$$8x + 2yy' - 12 = -4y' - 0$$
$$2yy' + 4y' = 12 - 8x$$
$$y'(2y + 4) = 12 - 8x$$
$$y' = \frac{12 - 8x}{2y + 4} = \frac{6 - 4x}{y + 2} = \frac{0}{1}$$

horiz tan, $m = 0$

$$6 - 4x = 0$$
$$4x = 6$$
$$x = \boxed{\frac{3}{2}}$$

**Example 3:** At what point(s) is the tangent line vertical on the curve given by $4x^2 + y^2 - 12x = -4y - 4$?

same y' as on EX 2.

$y' = \frac{6-4x}{y+2}$ ← For a vert tangent slope is ∞. Denom. will be 0.

$y+2=0$

$y=-2$

$4x^2+(-2)^2-12x=-4(-2)-4$

$4x^2-12x=0$

$x(4x-12)=0$

$x=0 \quad 4x-12=0$

$x=3$

pts → $\boxed{\begin{matrix}(0,-2)\\(3,-2)\end{matrix}}$

---

**Example 4:** Find the second derivative of *y* with respect to *x* in $x^2 = 10 + y^2$ by taking the derivative implicitly. Express the answer in terms of **only** *x* and *y*.

$\cancel{2}x = 0 + \cancel{2}yy'$

$\frac{x}{y} = y'$

$y' = xy^{-1}$

$y'' = x(-1)y^{-2}y' + y^{-1}(1)$

$y'' = \frac{-xy'}{y^2} + \frac{1}{y}$ ← sub in $y' = xy^{-1}$ from above

$y'' = -\frac{x\,xy^{-1}}{y^2} + \frac{1}{y}$

$y'' = \boxed{\frac{-x^2}{y^3} + \frac{1}{y}}$

**Assignment:** In problems in which derivatives are required, find them implicitly.

1. Find the equation(s) of the tangent line(s) to the curve given by $6y^2 = x$ at $x = 3$ .

2. Find the equation(s) of the normal line(s) to the curve given by $6y^2 = x$ at $x = 3$ .

3. At what *x* value(s) is the tangent to the curve given by $6y^2 = x$, horizontal?

4. At what *x* value(s) is the tangent to the curve given by $6y^2 = x$, vertical?

5. What are the equations for both the tangent and normal lines to the curve given by $-x^2 + y^2 = 15$ at $x = \sqrt{10}$?

---

6. Find y'' in terms of **only** *x* and *y* where $x^3 + xy = y$.

---

7. Find y'' in terms of **only** *x* and *y* where $4xy + x = 3$

8. What is the equation of the tangent line to the curve given by $y^2 + y - x = 2$ at (0, 1)?

---

9. At what point on the curve given by $xy = 1$ is the normal line a $45^o$ line with a positive slope?

---

10. Find the value of $d^2y/dx^2$ on the curve given by $xy=1$ at $(-1, -1)$.

**Unit 5: Lesson 03**

# Related rates

**Example 1:** Consider the classical example of a spherical balloon being inflated at the constant rate of 5π $cm^3$/sec. What is the rate at which the radius is changing when the radius is 112 cm?

$$V_{sphere} = \frac{4}{3}\pi r^3 \longrightarrow V' = \frac{4}{\cancel{3}}\pi \cancel{3} r^2 r'$$

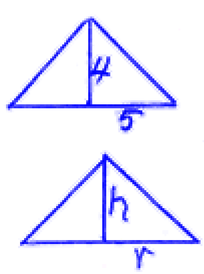
$$\frac{dV}{dt} = 5\pi = V' \text{ "given"} \qquad 5\cancel{\pi} = 4\cancel{\pi}(112)^2 r'$$

$$\frac{dr}{dt} = ? = r' \text{ "find"} \qquad \frac{5}{4(112)^2} = r'$$

$$r' = \boxed{9.9649 \times 10^{-5} \text{ cm/sec}}$$

---

**Example 2:** Sand is being poured onto a conical pile at the rate of 15 $ft^3$/min. The resulting friction of the sliding grains of sand forces the ratio of the height to the radius to be 4/5. What is the rate at which the height of the pile is changing when the height is 20 ft.




$$\frac{h}{r} = \frac{4}{5}$$

$$4r = 5h$$

$$r = \frac{5}{4}h$$

$$V_{cone} = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{5}{4}h\right)^2 h$$

$$V = \frac{1}{3}\pi \frac{25h^2}{16} h$$

$$V = \frac{25\pi}{48} h^3$$

$$\frac{dV}{dt} = V' = 15\, ft^3/min \text{ "Given"} \qquad V' = \frac{25\pi}{48} 3h^2 h'$$

$$\frac{dh}{dt} = h' = ? \text{ "Find"} \qquad 15 = \frac{75\pi}{48}(20)^2 h'$$

$$h' = \frac{15 \cdot 48}{75\pi\, 400}$$

$$= \boxed{.007639 \text{ ft/min}}$$

From the two examples above we can develop the following guidelines in solving a "related rate" problem:

- Determine what is "given" in terms of a rate (almost invariably a derivative w.r.t. time).
- Determine "what is being asked for"… typically a rate that is also a derivative w.r.t. time.
- Usually the resulting equation will have two independent variables. Get rid of one of the variables by:
    - finding an equation that relates the two variables, and
    - then eliminating one of the variables with a substitution from this equation.

**Assignment:**

1. A spherical hot-air balloon is losing its hot-air at the rate of 50 $ft^3$/min. What is the radial velocity of a bug clinging to the surface of the balloon when the balloon has a radius of 7 ft?

2. A cube's volume is increasing at the rate of 2.3 $m^3$/hr. At what rate is its surface area changing when a side of the cube is 3?

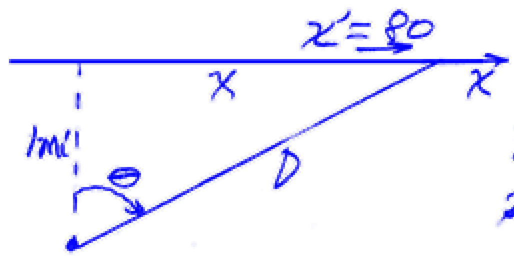
3. A conical water tank has a diameter of 8 meters at the top and a height of 6 meters. If water is leaking out at the rate of .5 $m^3$/min, find the rate at which the water level is falling when the radius of the water level is 2 meters?

4. A 12 ft ladder is leaned against a wall. The top of the ladder remains in contact with the wall while the bottom of the ladder is pulled horizontally along the floor with a speed of 1.5 ft/sec. With what speed is the top of the ladder descending when the top is 4 feet above the ground?

5. An object is moving along the curve described by $y = 4x^3$. Its shadow (projection) on the x-axis is observed to be moving along at the steady rate of 5m/sec. What will be the corresponding velocity of the shadow (projection) on the y-axis when $y = 32$?

# Unit 5: Lesson 04 More related rate problems

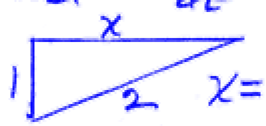
**Example 1:** Jake, a suspect in a recent jewel heist, is traveling east in his get-away car at 80 mph along a straight road. An FBI agent exactly one mile from the road is observing him through binoculars. How fast is the distance between the car and the agent changing when the distance between them is two miles?




$$x^2+1^2=D^2$$
$$\cancel{2}xx'=\cancel{2}DD'$$
$$D'=\frac{xx'}{D}=\frac{\sqrt{3}\,80}{2}$$
$$D'=\boxed{69.282\text{ mph}}$$

Given $\rightarrow \frac{dx}{dt}=x'=80$

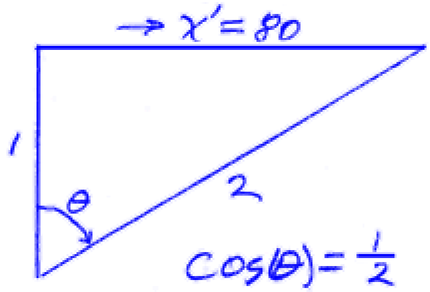
Find $\rightarrow \frac{dD}{dt}=D'=?$




$$x=\sqrt{2^2-1^2}$$
$$=\sqrt{3}$$

---

**Example 2:** Using the information from example 1, at what rate are the binoculars rotating when the distance between the agent and the car is two miles?




$$\cos(\theta)=\frac{1}{2}$$

$$\tan\theta=\frac{x}{1}$$
$$\sec^2(\theta)\,\theta'=x'$$
$$\theta'=x'/\sec^2(\theta)$$
$$\theta'=x'\cos^2(\theta)$$
$$\theta'=80\left(\frac{1}{2}\right)^2$$
$$\theta'=\boxed{20\text{ rad/hr.}}$$

**Assignment:**

1. A snowball is melting at the rate of 1.5 $cm^3$/min. How fast is the radius changing when the volume is 120 $cm^3$?

---

2. A 6 ft man is walking away from a 12 ft tall street light at 3 ft/sec. How fast is the length of his shadow increasing when he is 18 ft from the base of the light pole?

3. An object is moving along the curve given by $f(x) = 5x^3$ such that the projection of the object on the x-axis is constantly moving at 4 units/sec. What is the rate of change of the slope of the tangent line to the curve at the object's position when its x-position is 2?

---

4. An object is moving along the line $y = 2x + 1$ such that its projected speed on the x-axis is 4 m/sec. Find the rate of change of the distance between the object and the point (5,0) when the object's x-position is 2.

5. Two roads cross each other at right angles. A car on the north-south road is 1 mi north of the intersection and is approaching the intersection at 60 mph. Another car traveling 40 mph on the east-west road is 2 mi west of the intersection and headed west. How fast is the distance between the two cars changing?

---

6. A radar beam is following an airplane that is flying at an altitude of 1 mile. The plane flies directly over the radar site at 580 mph. At what angular rate is the radar antenna turning while following the plane when the shadow of the plane is 3 miles from the site? (The sun is directly overhead.)

# Unit 5: Cumulative Review

1. What is the derivative of sin($x^2$)?

   A. 2Sin(x)  B. sin($x^2$)2x  C. cos($x^2$)2x  D. 2cox(x)x

---

2. Assume f(0) = 5, f ’(0) = –1, f’’(0) = 2, and g(x) = cos(2x) f(x). What is g’(0)?

   A. –1  B. 0  C. 1  D. 10

---

3. Assume f(0) = π, f ’(0) = –1, f’’(0) = 2, and g(x) = sin( f(x) ). What is g’(0)?

   A. –1  B. 1  C. 0  D. π

---

4. The equation of the normal line to f(x) = 5x – π at x = 0 is:

   A. y – π = –.2(x – 1)  B. y – 0 = –5(x – π)  C. y + π = – x/5

   D. y = – x/5 + π

5. $\lim_{x \to 0} \frac{1 - \cos(x)}{x} = ?$

A. 0 B. undefined C. – 1 D. 1

---

6. $\lim_{x \to 0} \frac{\sin(x)}{x} = ?$

A. 0 B. undefined C. – 1 D. 1

---

7. $\lim_{x \to \infty} \frac{x^3 + 3x^2 + 1}{4x^3 - x} = ?$

A. 4 B. undefined C. 0 D. 1/4

---

8. Which limit does this table suggest?

A. $\lim_{x \to 0^-} f(x) = -11$

B. $\lim_{x \to 0^+} f(x) = -11$

C. $\lim_{x \to -11^-} f(x) = 0$

D. $\lim_{x \to 11^+} f(x) = 0$

| x | f(x) |
|---|---|
| –11.2 | –.1 |
| –11.18 | –.09 |
| –11.10 | –.009 |
| –11.02 | –.004 |
| –11.001 | –.001 |

---

9. What are all of the *x* values at which there are discontinuities in the function $f(x) = (x^2 - 4)/x(x^2 - x - 6)$?

A. 0, 2, 3 B. –3, 2 C. –2, 0, 3 D. –2, 3

10. Determine the value of *K* that will cause this function to be everywhere continuous.

$$f(x) = \begin{cases} Kx^3 & \text{if } x \le 2 \\ -3 & \text{if } x > 2 \end{cases}$$

A. 3/8

B. –3/8

C. 3

D. –8

---

11. Use $f'(c) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ to find the instantaneous rate of change of $f(x) = 6x^2 - x + 1$ at $x = -4$.

---

12. An object on the end of a spring oscillates back-and-forth according to f(t) = 5sin(4t) meters where *t* is measured in seconds. What are the velocity and acceleration as functions of time?

13. Explain why the $2^{nd}$ derivative of a linear function is 0.

---

14. Use $f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ to find f'(x) where f(x) =$\sqrt{x + 2}$ .

# Unit 5: Review

1. Differentiate implicitly to find y' where $x^2y = y\sin(x)$.

---

2. Differentiate implicitly to find y' where $\sin(x)\sin(y) = x^2$.

---

3. Find the x-position(s) at which the normal lines to the curve given by $xy^2 = 1$ are parallel to $4x + 4y = 7$.

4. Differentiate implicitly to find y'' in terms of x and y where $x^{1/2} + y^{1/2} = x$.

---

5. Consider a circle having a radius of 3 and center at (2, 4). Locate the point(s) at which the tangent line(s) to this circle have a slope equal to the slope of the line connecting the origin and the center of the circle. (Differentiate implicitly.)

---

6. Find g' where g(x) =sin( sin(x) )

7. An object is moving left-to-right along the curve $y = (4/3)x^3$. At what point(s) on the curve are the *x* and *y* components of its velocity equal?

---

8. Astronomers are interested in dying stars and their rate of shrinkage. To understand the problems they need to solve, consider a star that is losing volume at the rate of 500 $km^3$/sec. What is the rate at which the surface area is changing when the radius is $2 \text{ X } 10^4$ km? (Assume that the star is spherical.)

---

9. What is dg/dm evaluated at m = 3 radians where *g* is a function of *m* such that $g^2m^2 + 3\sin(m) = 7$? (Assume that g(3) = –4.)

10. Assuming that *y* is a function of *x*, find y' in terms of *x* and *y* where $1/(x + y^2) = y^{1/2}$.

# Calculus, Unit 6

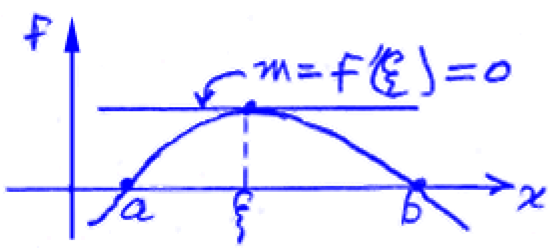
## Rolle's Theorem and the Mean Value Theorem
## First and second derivative tests
## Critical values

# Unit 6: Lesson 01 Rolle's Theorem and the Mean Value Theorem

**Rolle's Theorem** as applied to a function f(x) in the closed interval $a \le x \le b$ states that if:

- $f(a) = f(b) = 0$,
- f(x) is continuous in $a \le x \le b$, and
- f(x) is differentiable in $a < x < b$, then

There is at least one point ξ (Greek letter xi) in the open interval $a < x < b$ such that $f'(\xi) = 0$.

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**Example 1:** Establish that all three requirements of Rolle's Theorem are satisfied for the function $f(x) = x^2 - x - 12$ in the closed interval $-3 \le x \le 4$. Then find the ξ value(s) predicted by the theorem.

$f(-3) = (-3)^2 - (-3) - 12 = 9 + 3 - 12 = 0$ ✓

$f(4) = (4)^2 - 4 - 12 = 16 - 4 - 12 = 0$

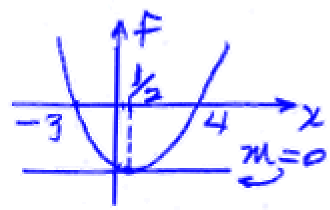
Polynomial ⇝ cont. everywhere ✓

Polynomial ⇝ diff everywhere ✓

$f' = 2x - 1 = 0$

$2x = 1$

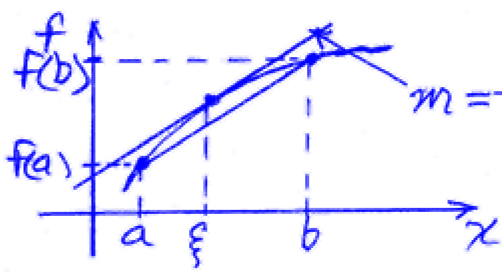
$\xi = x = \boxed{\frac{1}{2}}$



**The Mean Value Theorem** as applied to a function f(x) in the closed interval $a \le x \le b$ states that if:

- f(x) is continuous in the closed interval $a \le x \le b$, and
- f(x) is differentiable in the open interval $a < x < b$, then

There is at least one point ξ in the open interval $a < x < b$ such that $f'(\xi) = (f(b) - f(a))/(b - a)$.




$$m = \frac{f(b) - f(a)}{b - a}$$

---

**Example 2:** Establish that both requirements of The Mean Value Theorem are satisfied for the function $f(x) = x^2$ in the closed interval $1 \le x \le 4$. Then find the ξ value(s) predicted by the theorem.

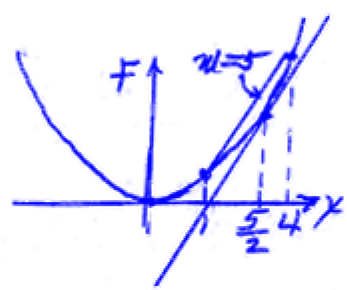
polynomial → everywhere cont. ✓

polynomial → everywhere diff. ✓

$$m = \frac{f(4) - f(1)}{4 - 1} = \frac{16 - 1}{3} = \frac{15}{3} = 5$$

$$f'(x) = 2x = 5$$

$$x = \boxed{\frac{5}{2}} = \xi$$



**Assignment:**

1. Show that the function $f(x) = 2x^2 + 27x - 14$ satisfies the conditions of Rolle's Theorem in the interval [–14, .5] and then find the value(s) of ξ predicted by the theorem.

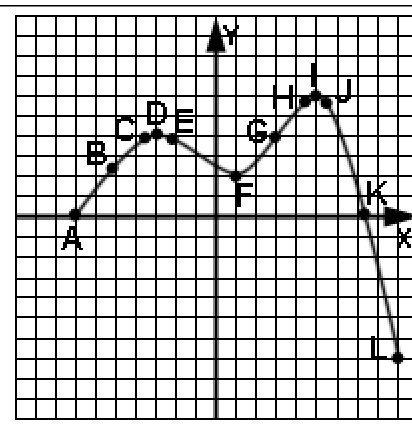
2. Show that the function $f(x) = x^3 + 2x^2 - 3x$ satisfies the conditions of Rolle's Theorem in the interval $0 \le x \le 1$ and then find the value(s) of ξ predicted by the theorem.

3. Show that the function $f(x) = \sqrt{x}$ satisfies the conditions of The Mean Value Theorem in the interval $16 \le x \le 25$ and then find the value(s) of ξ predicted by the theorem.

4. Show that the function f(x) = 1/(x – 6) satisfies the conditions of the Mean Value Theorem in the interval *6.1 ≤ x ≤ 10* and then find the value(s) of ξ predicted by the theorem.

5. Consider the function f(x) = sin(x) in the closed interval [0, 3π/2]. Find the value(s) predicted by the Mean Value Theorem in the corresponding open interval.

6. In the closed function interval [A, L] depicted here, determine from the listed points, the point(s) that are predicted by Rolle's Theorem in its corresponding open interval.

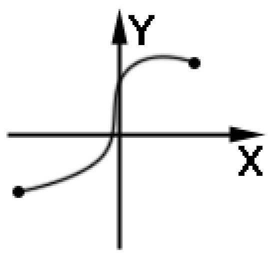


7. For the function interval [F, L] in problem 6, determine from the listed points the point(s) predicted by the Mean Value Theorem.

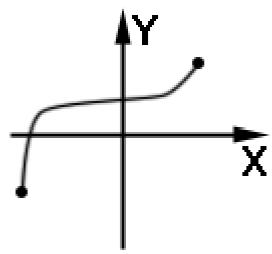
8. For the function interval [A, I] in problem 6, determine from the listed points the point(s) predicted by the Mean Value Theorem.

9. Which interval(s) will yield exactly two values satisfying the Mean Value Theorem? (Assume each function is continuous at the end points of the intervals shown.)

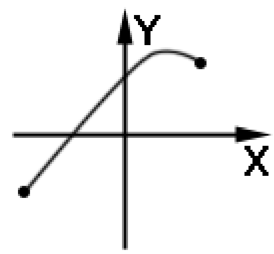
A.



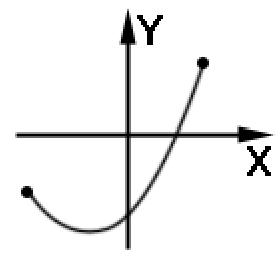

B.




C.

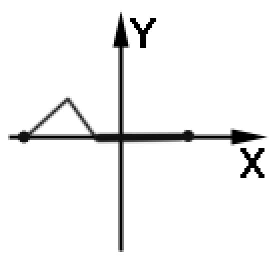



D.

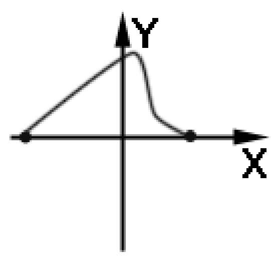



10. Which interval(s) will yield only one value satisfying Rolle's Theorem? (Assume each function is continuous at the end points of the intervals shown.)

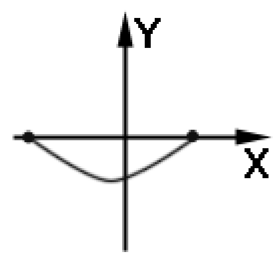
A.



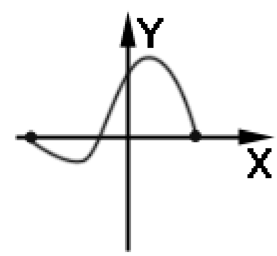

B.




C.




D.




11. What value of *x* satisfies the Mean Value Theorem $f(x) = -x^3 + x$ over the interval $[-2, 1]$?

# Unit 6: Lesson 02 — First derivative test: increasing/decreasing intervals
## Critical values

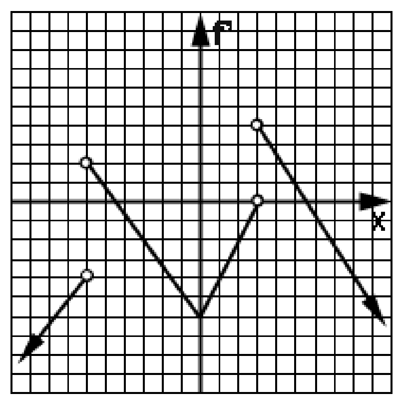
If a function f(x) is continuous over an interval and f ’(x) > 0 then f(x) is **increasing** in that interval.

If a function f(x) is continuous over an interval and f ’(x) < 0 then f(x) is **decreasing** in that interval.

A **local maximum** occurs when the derivative switches from positive to negative.

A **local minimum** occurs when the derivative switches from negative to positive.

**Example 1:** The adjacent picture shows the **derivative** f ’(x) of the function f(x). For each question below, justify your answer.




A. Over what interval(s) is f(x) increasing?
*(-6, -4.5), (3, 5.5); f’ is positive*

B. Over what interval(s) is f(x) decreasing?
*(-∞,-6), (-4.5, 3), (5.5, ∞) ; f’ is negative*

C. At what *x* value(s) does f(x) have a local maximum?
*-4.5, and 5.5 …derivative switches from positive to negative.*

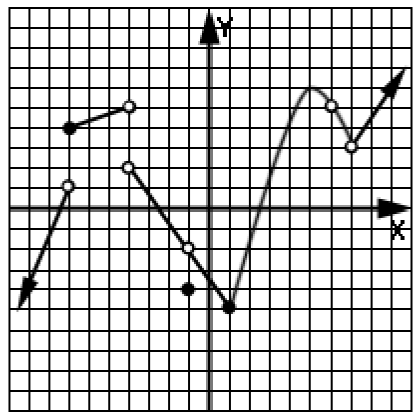
D. At what *x* value(s) does f(x) have a local minimum?
*-6, 3 …derivative switches from negative to positive.*

**Critical values:**

- If f(x) is defined at *c*, and
- if either
  - o f ’(c) = 0, or
  - o f ’(c) does not exist, then

*c* is said to be a critical x-value of f(x).

**Example 2:** For the following x-values of f(x) (shown to the right), state if it is a critical x-value or not. In each case justify your answer.




A. x = –7
*cv… y is defined and derivative doesn’t exist.*

B. x = –4
*not a cv… y does not exist here.*

C. x = –1
*cv… y is defined and derivative doesn’t exist*

D. x = 1
*cv… y is defined and derivative doesn’t exist (cusp).*

E. x = 5
*cv… y is defined and derivative = 0.*

F. x = 6
*not a cv… y does not exist here.*

G. x = 7
*not a cv… y does not exist here.*

**Example 3:** Determine the critical x-values for the function f(x) = $-6x^2 + 2x + 1$

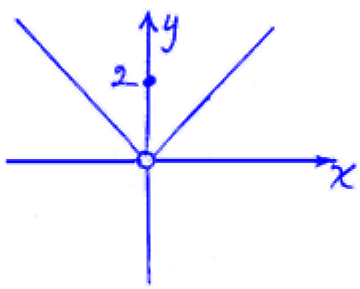
$$f'(x) = -12x + 2 = 0$$
$$-12x = -2$$
$$x = \boxed{\frac{1}{6}} \leftarrow \text{cv}$$

$f\left(\frac{1}{6}\right)$ is defined here
and $f' = 0$ here

---

**Example 4:** Determine the critical x-value(s) for the function defined by:

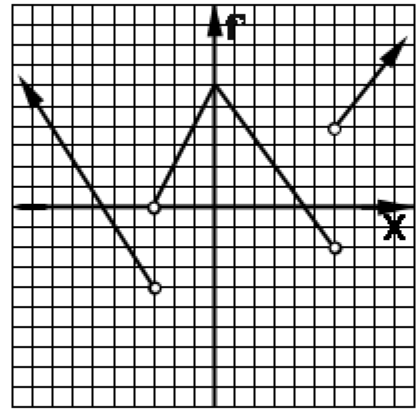
$$f(x) = f(x) = \begin{cases} -x, & x < 0 \\ 2, & x = 0 \\ x, & x > 0 \end{cases}$$




$f(0)$ exists
$f'(0)$ does not exist
(disc at a cusp)

cv → $x = 0$

**Assignment:**




1. The adjacent picture shows the **derivative** f ′(x) of the function f(x). For each question below, justify your answer.

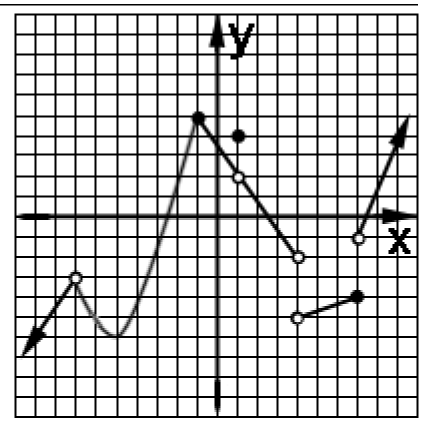
A. Over what interval(s) is f(x) increasing?

B. Over what interval(s) is f(x) decreasing?

C. At what *x* value(s) does f(x) have a local maximum?

D. At what *x* value(s) does f(x) have a local minimum?

---




2. For each x-value of the function shown to the right, state if it is a critical x-value or not. Justify your answer.

A. x = –7

B. x = –5

C. x = –1

D. x = 1

E. x = 4

F. x = 7

In the following problems, determine for the interval given (justify all answers):

(a) All of the critical values

(b) Local minima and maxima

(c) Intervals of increase and/or decrease

(d) The value of ξ that satisfies the Mean Value Theorem.

---

3. $f(x) = \cos(x) + \cos^2(x)$ $[0, 2\pi]$

4. $f(x) = \cos(x) - x$ $[0, 2\pi]$

5. $f(x) = 4x^{1/3}$ $[-27, 27]$

## Unit 6: Lesson 03 Local and absolute extrema

**Extreme Values Type #1** (look at critical points):

**Local maximum** (relative maximum):

A function has a local maximum at x = c if there is an interval containing *c* in its interior such that in that interval f(c) is greater than all other values of f(x).

**Derivative test:** At *c* the derivative changes from positive to negative.

**Local minimum** (relative minimum):

A function has a local minimum at x = c if there is an interval containing *c* in its interior such that in that interval f(c) is less than all other values of f(x).

**Derivative test:** At *c* the derivative changes from negative to positive.

**Example 1:** Draw an example of a function having two local maxima and two local minima in the interval [a, b].


---

**Example 2:** Find the local extrema for the function $f(x) = 6x^4 - 40x^3 - 72x^2$.

$f'(x) = 24x^3 - 120x^2 - 144x = 24x(x^2 - 5x - 6)$

$f'(x) = 24x(x-6)(x+1) = 0$

$x = 0,\ 6,\ -1$

$f' \to$ — $-1$ + $0$ — $6$ +

$\boxed{\text{min at } x = -1 + 6}$ Der. goes from – to +

$\boxed{\text{max at } x = 0}$ Der. goes from + to –

**Extreme Values Type #2** (look at critical points and end points)

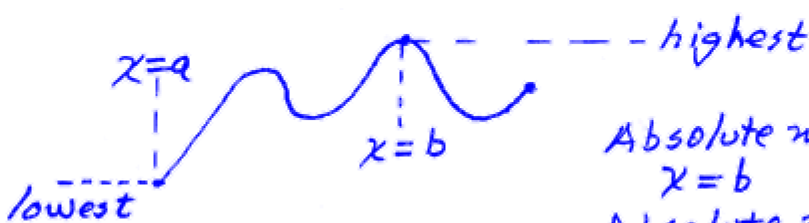
**Absolute maximum**

In an interval the function f(x) has an absolute maximum at x = c if f(c) is greater than or equal to **all** other f(x) in that interval.

**Absolute minimum**

In an interval the function f(x) has an absolute minimum at x = c if f(c) is less than or equal to **all** other f(x) in that interval.

**Example 3:** Draw an example of a function with labeling that emphasizes the absolute maximum and minimum.




Absolute max at $x=b$

Absolute min at $x=a$

---

**Example 4:** Find the absolute maximum and minimum of $f(x) = -x^3 - 4x^2 + 1$ over the interval [–5, 1].

$f'(x) = -3x^2 - 8x = 0$

cv's: → $-2.\bar{6}$ + 0

end pt

$f(-5) = 26$ ← largest

$f(-2.\bar{6}) = -8.4815$ ← smallest

$f(0) = 1$

$f(1) = -4$

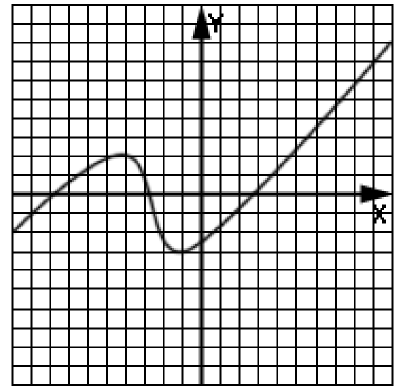
end point

Absolute max at $(-5, 26)$

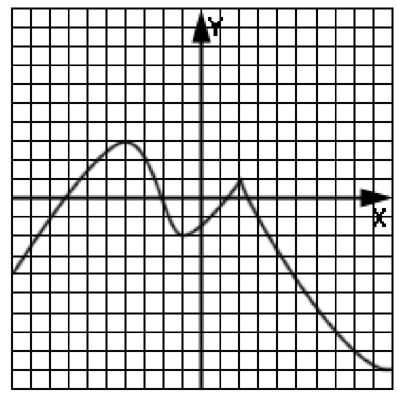
Absolute min at $(-2.\bar{6}, -8.4815)$

**Assignment:**

1. Identify the x-values at which there are local extrema and absolute extrema over the interval [-10, 10].




2. Identify the x-values at which there are local extrema and absolute extrema over the interval [-10, 10].




3. Find all local and absolute extrema for $f(x) = x^3 - 27x$ in the interval [–7, 5].

4. Find all local and absolute extrema for f(x) = sec(x) in the interval [2π/3, 5π/6].

---

5. Find all local and absolute extrema for f(x) = 2/(x + 4) in the interval [–5, 0].

---

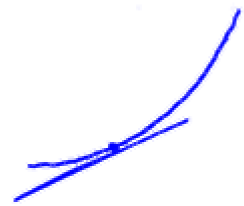
6. Find all local and absolute extrema for:

$$f(x) = \begin{cases} -x^2 & 2 < x \leq 4 \\ x + 1 & 0 < x \leq 2 \end{cases}$$

# Unit 6: Lesson 04 Second derivative test: Concavity

**Concave upward:**

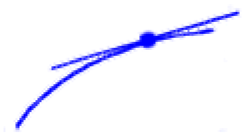
If the graph of f(x) is everywhere **above** all of the tangents to the curve in an interval, then the curve is said to be concave upward.



**$f''(x) > 0$** everywhere in the interval.

**Concave downward:**

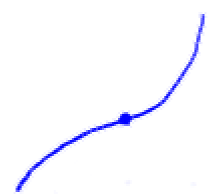
If the graph of f(x) is everywhere **below** all of the tangents to the curve in an interval, then the curve is said to be concave downward.



**$f''(x) < 0$** everywhere in the interval.

**Point of inflection:**

The point at which a curve **changes its concavity** is called a point of inflection.



**$f''(x)$ changes sign.**

**Example 1:** Find the intervals of concavity and points of inflection of $f(x) = x^4 + 6x^3$.

$f'(x) = 4x^3 + 18x^2$

$f''(x) = 12x^2 + 36x = 0$

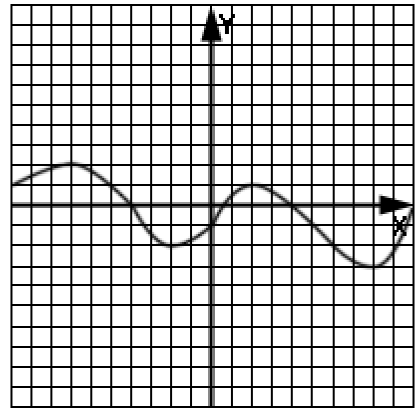
$x(12x + 36) = 0$ $\quad$ $f''$: $+$ $\;-3\;$ $-$ $\;0\;$ $+$

$x = \boxed{0}$ $\quad$ $12x + 36 = 0$

pts of inflection $\rightarrow$ $x = \boxed{-3}$

$$\boxed{\begin{array}{ll} \text{cc up:} & (-\infty, -3), (0, \infty) \\ \text{cc dwn:} & (-3, 0) \end{array}}$$

**Example 2:** The graph to the right is the function f(x). Fill in the table using "+", "– " , or "0" to indicate the value/sign of the functions at each x-position.




| | f(x) | f '(x) | f ''(x) |
|---|---|---|---|
| x= –10 | + | + | – |
| x = –7 | + | 0 | – |
| x= –4 | 0 | – | 0 |
| x = –2 | – | 0 | + |
| x = 0 | – | + | 0 |
| x = 2 | + | 0 | – |
| x = 4 | 0 | – | – |
| x = 6 | – | – | 0 |
| x = 8 | – | 0 | + |
| x = 9 | – | + | + |
| x = 10 | 0 | + | + |

**Assignment:**

1. What is the sign of f '' in an interval of f(x) where f is concave upward?

2. What is the sign of f '' in an interval of f(x) where f is concave downward?

3. Find the intervals of concavity and points of inflection of $f(x) = 2x^4 - 6x^2$.

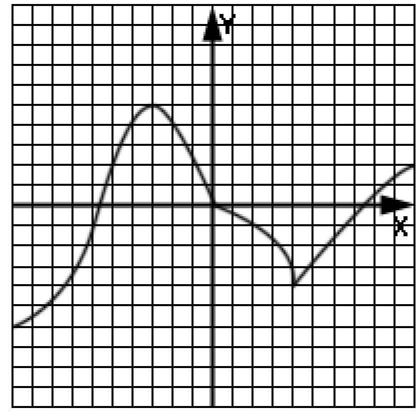
4. Find the intervals of concavity and points of inflection of $f(x) = x^2 - 10$.

5. Find the intervals of concavity and points of inflection of $f(x) = 7x + 11$.

6. Find the intervals of concavity and points of inflection of $f(x) = x^2 + 3\sin(x)$ over the interval $[0, 2\pi]$.

7. $f(x) = 2x^3 - 3x^2 - 36x + 7$

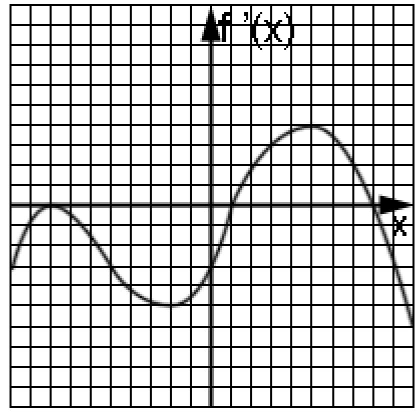
8. The graph to the right is the function f(x). Fill in the table using "+", "– " , "0", or "undef" to indicate the value/sign of the functions at each x-position.



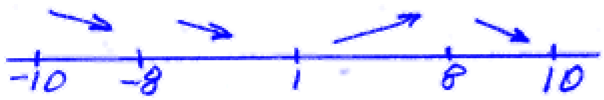

| | **f(x)** | **f '(x)** | **f ''(x)** |
|---|---|---|---|
| x= –10 | | | |
| x = –5.8 | | | |
| x= –3 | | | |
| x = –2 | | | |
| x = 0 | | | |
| x = 2 | | | |
| x = 4 | | | |
| x = 8 | | | |
| x = 9 | | | |

# Unit 6: Lesson 05 Graphs relating f(x), f '(x), and f ''(x)

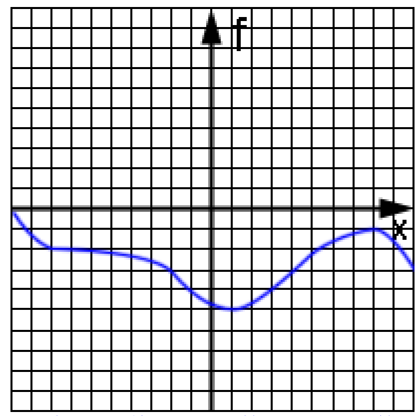
**Example:** The derivative, f ’(x), of the function f(x) is shown to the right. The function is defined over the interval shown, [–10, 10]. Answer the following questions about *f* (always providing justification) and then provide a sketch of the function f(x).




A. Find the intervals of increase/decrease.
*decr [-10, -8), (-8, 1), (8, 10], neg derv*
*incr (1, 8), pos derv*




B. Find the local maxima and minima.
*max at x = 8, derv changes from pos to neg.*
*min at x = 1, derv changes from neg to pos.*


*Lift or lower this graph by any amount.*

C. Find the intervals of concavity and points of inflection.

*ccup: [-10, -8), (-2, 5), f’ is increasing*

*ccdwn: (-8, -2), (5, 10], f’ is decreasing*

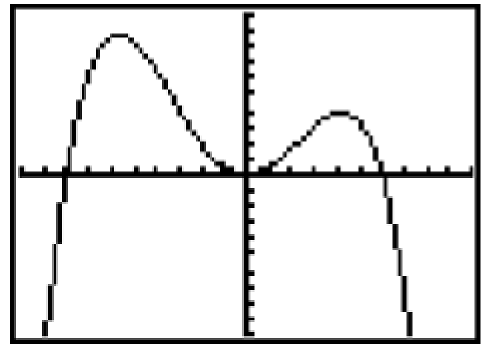
*pts of infl at -8, -2, and 5 f’ changes from incr to decr or decr to incr*

**Assignment:**

| | |
|---|---|
| 1. What is the justification for claiming that *f* is increasing on an interval? | 2. What is the justification for claiming that *f* is decreasing on an interval? |
| 3. What is the justification for claiming that *f* has a local maximum at x = c? | 4. What is the justification for claiming that *f* has a local minimum at x = c? |
| 5. In terms of f ′′ what is the justification for claiming that *f* is concave upward over an interval? | 6. In terms of f ′′ what is the justification for claiming that *f* is concave downward over an interval? |
| 7. In terms of f ′ what is the justification for claiming that *f* is concave upward over an interval? | 8. In terms of f ′ what is the justification for claiming that *f* is concave downward over an interval? |

9. In terms of f ′ what is the justification for claiming that *f* has a point of inflection at x = c?

10. The derivative, f ′(x), of the function f(x) is shown to the right. The function is defined over the interval shown, [–10, 10]. Answer the following questions about *f* (always providing justification) and then provide a sketch of the function f(x).



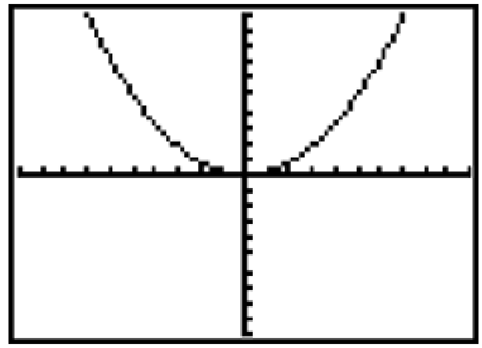
Shown above is f ′(x) defined over the interval [–10, 10].

A. Find the intervals of increase/decrease.

B. Find the local maxima and minima.

C. Find the intervals of concavity and points of inflection.

11. The derivative, f ’(x), of the function f(x) is shown to the right. The function is defined over the interval shown, [–10, 10]. Answer the following questions about *f* (always providing justification) and then provide a sketch of the function f(x).



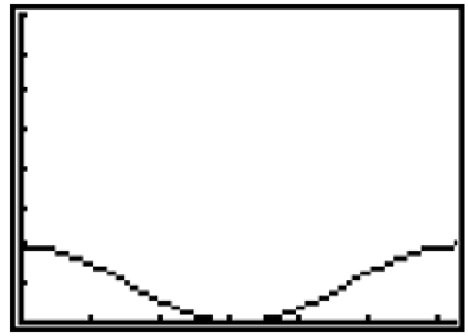
Shown above is f ’(x) defined over the interval [–10, 10].

A. Find the intervals of increase/decrease.

B. Find the local maxima and minima.

C. Find the intervals of concavity and points of inflection.

12. The derivative, f ’(x), of the function f(x) is shown to the right. The function is defined over the interval shown, [0, 2π]. Answer the following questions about *f* (always providing justification) and then provide a sketch of the function f(x).



Shown above is f ’(x) defined over the interval [0, 2π].

A. Find the intervals of increase/decrease.

B. Find the local maxima and minima.

C. Find the intervals of concavity and points of inflection.

# Unit 6: Cumulative Review

1. Which one of the following is a correct representation of the derivative of f(x) with respect to *x*?

A. $\lim_{\Delta x \to \infty} \frac{f(x)-f(x+\Delta x)}{\Delta x}$ B. $\lim_{\Delta x \to \infty} \frac{f(\Delta x)-f(x)}{\Delta x}$

C. $\lim_{\Delta x \to 0} \frac{-f(x)+f(x+\Delta x)}{\Delta x}$ D. $\lim_{\Delta x \to 0} \frac{f(x)-f(x+\Delta x)}{x+\Delta x}$

---

2. Which one of the following is correct?

A. $\lim_{x \to 0} \frac{1-\sin(x)}{x} = 1$ B. $\lim_{x \to 0} \frac{\cos(x)}{x} = 1$

C. $\lim_{x \to 0} \frac{1+\cos(x)}{x} = 1$ D. $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ E. None of these

---

3. Which one of the following is correct?

A. $\lim_{x \to 0} \frac{x}{\sin(x)} = 1$ B. $\lim_{x \to 0} \frac{\cos(x)}{x} = 1$ C. $\lim_{x \to 0} \frac{1-\cos(x)}{x} = 1$

D. $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ E. More than one of these

---

4. Consider a circle that is tangent to the curve $y = x^2$ at x = –2. What is the equation of the line passing through both this point of tangency and the center of the circle?

5. Find the derivative of *y* w.r.t. *x* in terms of *x* and y from the equation $yx - \sin(y) = x^2$.

---

6. Find p' w.r.t. *x* where $p = \sin(\cos(p)) + x$.

---

7. A spherical bladder in a fish is spontaneously inflating so as to enable the fish to rise toward the surface. If the rate of increase of the radius is 1.2 cm/min, what is the corresponding rate of increase of the volume of the bladder when its diameter is 3 cm.?

8. Which of the following is the derivative of $-\csc^2(3x)$?

A. $2\csc^2(3x)\cot(3x)$ B. $6\csc^2(3x)\cot(3x)$

C. $-6\csc^2(3x)\cot(3x)$ D. $18\csc(3x)\cot(3x)$ E. $18\csc^2(3x)\cot(3x)$

---

9. Which of the following is a correct representation of how to obtain the derivative of f(x) at x = c?

A. $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ B. $\lim_{x\to 0}\frac{f(x)-f(c)}{x-c}$ C. $\lim_{x\to c}\frac{f(c)-f(x)}{c-x}$

D. $\lim_{x\to c}\frac{f(x-c)-f(c)}{x-c}$ E. More than one of these

---

10. Assume that $f(4) = -5$, $f'(4) = 11$, $g(4) = 9$, and $g'(4) = 1$. Also assume that $f(-5) = 2$, $f'(-5) = 10$, $g(-5) = 7$; and $g'(-5) = -2$. If $p(x) = g(f(x))$, find the value of $p'(4)$.

A. –110 B. 20 C. –22 D. 22 E. 110

11. Using the fundamental definition of the general derivative,

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h},$$

find the derivative of $f(x) = \dfrac{2}{2-\sqrt{x}}$ :

# Unit 6: Review

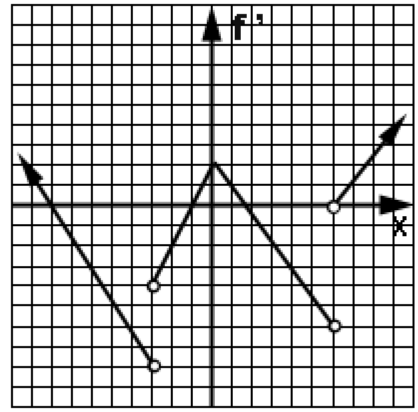
1. What are the three conditions of Rolle's Theorem that must be met in order to guarantee the existence of ξ as predicted by Rolle's Theorem over the interval [a, b] on the function f(x)?

2. What are the two conditions of the Mean Value Theorem that must be met in order to guarantee the existence of ξ as predicted by the theorem over the interval [a, b] on the function f(x)?

3. Consider the function $f(x) = x^2 - 2x - 99$ over the interval [–9, 11]. Show that the conditions of Rolle's Theorem are met over this interval and then find the value of ξ predicted by the theorem.

4. Consider the function $f(x) = 4\sin(x) + x$ over the interval [0, π]radians. Show that the conditions of the Mean Value Theorem are met over this interval and then find the value of ξ predicted by the theorem.

5. The adjacent picture shows the **derivative** f ′(x) of the function f(x). For each question below, justify your answer.




A. Over what interval(s) is f(x) increasing?

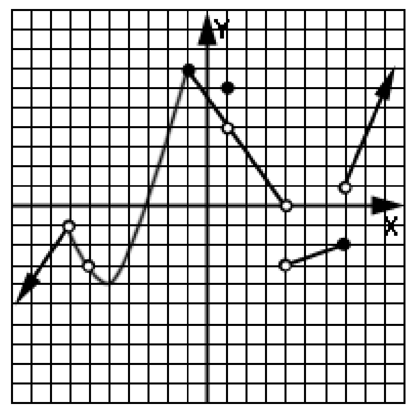
B. Over what interval(s) is f(x) decreasing?

C. At what *x* value(s) does f(x) have a local maximum?

D. At what *x* value(s) does f(x) have a local minimum?

---

6. For each *x* value of the function shown to the right, state if it's a critical value or not. Justify your answers.




A. x = –7

B. x = –6

C. x = –5

D. x = –1

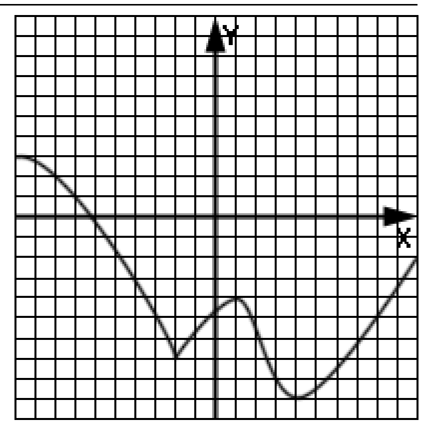
E. x = 1

F. x = 4

G. x = 7

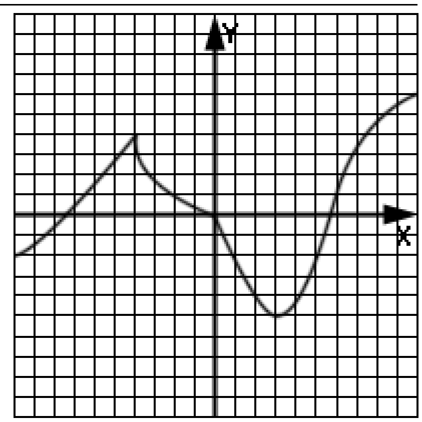
7. Consider the function $f(x) = 11\sin(2x)$ over the interval $(0, 3\pi/4]$. Find (a) the intervals of increase and decrease, (b) the critical points, (c) local minima and maxima, and (d) the absolute minimum and maximum. Justify each answer.

8. Identify the x-values at which there are local extrema and absolute extrema over the interval [-10, 10].



9. Find the intervals of concavity and points of inflection of $f(x) = .25x^4 - x^3 - 12x$.

10. The graph to the right is the function f(x). Fill in the table using "+", "– " , "0", or undef to indicate the value/sign of the functions at each x-position.




| | **f(x)** | **f '(x)** | **f "(x)** |
|---|---|---|---|
| x= –10 | | | |
| x = –9 | | | |
| x = –7.2 | | | |
| x= –4 | | | |
| x = –2 | | | |
| x = 0 | | | |
| x = 3 | | | |
| x = 5.7 | | | |
| x = 7 | | | |
| x = 10 | | | |

11. For the function, f(x), in problem 10, what is the absolute maximum function value and at what *x* value does it occur?

12. For the function, f(x), in problem 10, what is the absolute minimum function value and at what *x* value does it occur?

# Calculus, Unit 7

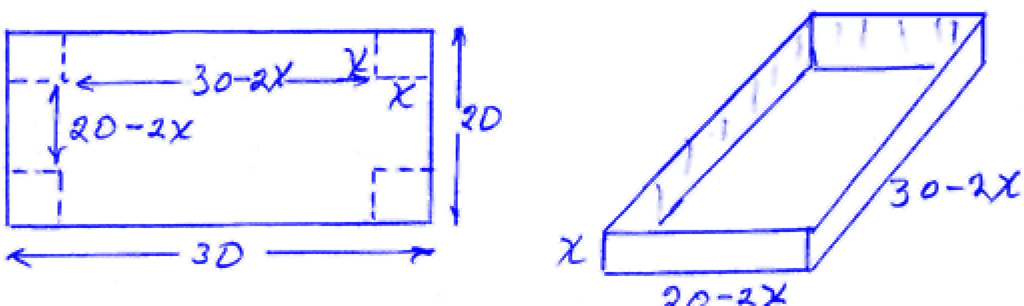
# Optimization (maximizing & minimizing)

**Unit 7: Lesson 01** **Optimization problems**

Optimization typically involves finding the maximum or minimum of some quantity:

- Decide on what variable is to be optimized.
- Create a function in which that variable is in terms of just one other variable.
- Find the critical points of the function.
- Find the absolute maximum or minimum value (as appropriate).

**Example 1 (maximizing):** A rectangular sheet of metal 20 in X 30 in will have small squares cut out of each corner and then discarded. The four "sides" are then folded up (and the seams welded) so as to form a lidless rectangular pan for cooking cane syrup. What size squares should be cut out of each corner so as to maximize the volume of the syrup pan? What is the maximum volume?




$V = x(30-2x)(20-2x)$

$V = 600x - 100x^2 + 4x^3$

$V' = 600 - 200x + 12x^2 = 0$

$3x^2 - 50x + 150 = 0$

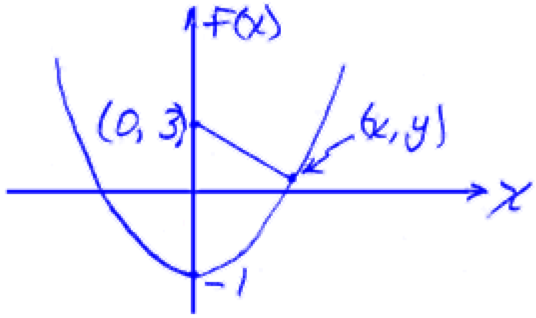
$x \approx \boxed{3.9237}$ in $;\ \cancel{12.743} \leftarrow$ outside the domain

$V = 3.9237(30 - 2\cdot 3.9237)(20 - 2\cdot 3.9237)$

$V = \boxed{1056.3059 \text{ in}^3}$

Domain: $20 - 2x > 0$ $\therefore$ $x < 10$

**Example 2 (minimizing):** Consider the parabola $f(x) = x^2 - 1$. What is(are) the point(s) on the parabola closest to the point (0, 3)?




$$d = \sqrt{(x-0)^2 + (y-3)^2} = \left(x^2 + (y-3)^2\right)^{1/2}$$

$$y = x^2 - 1$$

$$d = \left(x^2 + (x^2-1-3)^2\right)^{1/2} = \left(x^2 + (x^2-4)^2\right)^{1/2}$$

$$d = \left(x^2 + x^4 - 8x^2 + 16\right)^{1/2}$$

$$d = \left(x^4 - 7x^2 + 16\right)^{1/2}$$

$$d' = \tfrac{1}{2}\left(x^4 - 7x^2 + 16\right)^{-1/2}\left(4x^3 - 14x\right) = 0$$

$$4x^3 - 14x = 0$$

$$x(4x^2 - 14) = 0$$

$$\cancel{x = 0} \qquad x^2 = \frac{14}{4} = 7/2$$

$$x = \pm\sqrt{7/2}$$

$$d(0) = 4$$

$$d(\sqrt{7/2}) = 1.936 \leftarrow$$ Absolute mins

$$d(-\sqrt{7/2}) = 1.936$$

$$f(\sqrt{7/2}) = \frac{7}{2} - 1 = \frac{5}{2} \rightarrow \boxed{\left(\sqrt{\tfrac{7}{2}}, \tfrac{5}{2}\right) \quad \left(-\sqrt{\tfrac{7}{2}}, \tfrac{5}{2}\right)}$$

$$f(-\sqrt{\tfrac{7}{2}}) = \frac{7}{2} - 1 = 5/2$$

*This gives a local maximum as evidenced by the deriv changing from pos to neg here.*

**Assignment:**

1. What is the shortest distance from (2, 0) to the function $f(x) = x^2$?

2. What are the dimensions of a rectangle giving the maximum area when a diagonal of the rectangle must be 30?

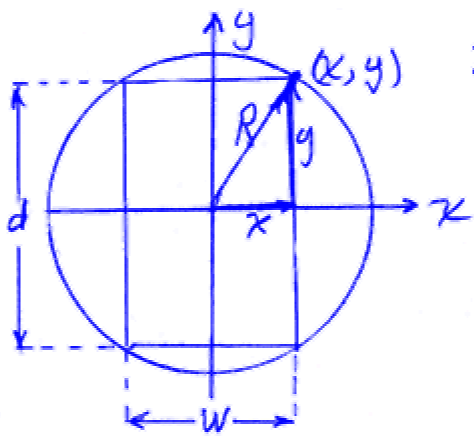
3. A rectangle is to be inscribed inside a semi-circle of radius 5. What are the dimensions of the rectangle giving the largest area?

4. Find two positive numbers whose product is 220 and whose sum is as small as possible.

5. Find two positive numbers whose sum is 10. The sum of their cubes is to be a minimum.

# Unit 7: Lesson 02 More optimization problems

**Example 1:** A saw mill finds that the stiffness of a rectangular beam is directly proportional to the width of the beam and to the cube of its depth. What are the dimensions of a rectangular beam of maximum stiffness that could be cut from a circular cross-section log of radius R?




$$x^2+y^2=R^2$$

$$y=\sqrt{R^2-x^2}$$

constant of proportionality

$$S=kwd^3$$

$$S=k(2x)\cdot(2y)^3$$

$$S=k2x\cdot 8(R^2-x^2)^{3/2}$$

$$S=16kx(R^2-x^2)^{3/2}$$

$$S'=\cancel{16K}x\,\tfrac{3}{2}(R^2-x^2)^{1/2}(-2x)+(R^2-x^2)^{3/2}\cancel{16K}=0$$

$$\frac{(R^2-x^2)^{3/2}}{(R^2-x^2)^{1/2}}=\frac{3x^2\cancel{(R^2-x^2)^{1/2}}}{\cancel{(R^2-x^2)^{1/2}}}$$

$$R^2-x^2=3x^2$$

$$R^2=4x^2$$

$$x^2=\frac{R^2}{4}$$

$$x=\sqrt{R^2/4}$$

$$x=R/2$$

$$w=2x=\boxed{R}$$

$$y=\sqrt{R^2-x^2}$$

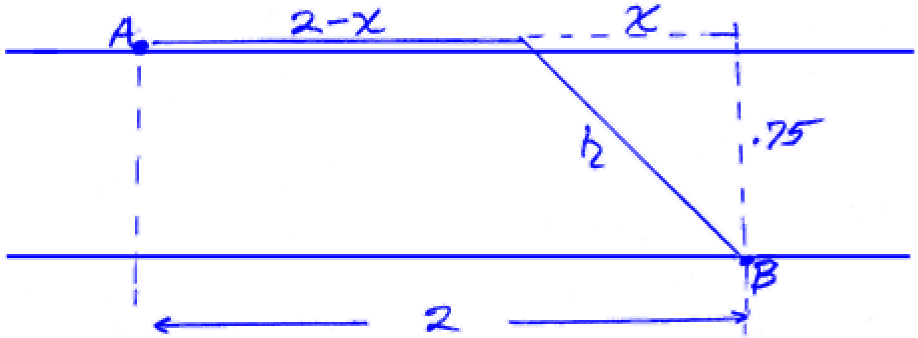
$$y=\sqrt{R^2-(R/2)^2}$$

$$y=\sqrt{\frac{4R^2}{4}-\frac{R^2}{4}}$$

$$y=R\sqrt{\frac{3}{4}}=\frac{R\sqrt{3}}{2}$$

$$d=2y=\boxed{R\sqrt{3}}$$

**Example 2:** A construction company is given a contract to build a pipeline from point A on the bank of river to point B on the opposite bank and 2 miles downstream. If the river is .75 miles wide, what is the cheapest construction cost to build the pipeline if it costs 1 million dollars per mile for a pipeline built on dry land and 2 million dollars for each mile in water?




$h^2 = x^2 + (.75)^2$

$h = (x^2 + 9/16)^{1/2}$ $\qquad$ c = cost in millions

$c = (2-x)1 + (x^2 + 9/16)^{1/2}(2)$

$c' = -1 + (1/2)(x^2 + 9/16)^{-1/2} 2x \cdot 2 = 0$

$\dfrac{2x}{(x^2+9/16)^{1/2}} = 1$

$(x^2 + 9/16)^{1/2} = 2x$

$x^2 + 9/16 = 4x^2$

$3x^2 = \frac{9}{16}$

$x^2 = \frac{3}{16}$ $\qquad$ $c\left(\frac{\sqrt{3}}{4}\right) = \boxed{3.299 \text{ million}}$

$x = \frac{\sqrt{3}}{4}$

**Assignment:**

1. A gardener wants to enclose 600 $ft^2$ with a rectangular fence. There is to be one cross-fence subdividing the area into two halves. So as to gain access to the two halves, there needs to be two 5 ft gaps at which gates will be placed. Additionally, the gardener wants a similar gap in the cross fence. What is the minimum number of feet of fencing that will do the job? What are the outside dimensions of the rectangular area?

2. A long straight piece of wire of length L is to be cut into two pieces. One piece is to be bent into the shape of an equilateral triangle. The other is to be bent into the shape of a square. Where should the wire be cut in order to maximize the sum of the areas of the square and triangle?

3. An isosceles triangle is to be inscribed in a circle of radius 1. What are the dimensions of the triangle that will result in a triangle of maximum area?

4. Find two integers whose product is 36 and whose sum is as large as possible.

5. A rectangular commercial building is to have 1200 $m^2$ of floor space. The siding that faces the street will be constructed of material costing $400 per meter while the other three sides will be made with cheaper material costing only $150 per meter. What should be the dimensions of the building in order to minimize the cost of the siding?

# Unit 7: Lesson 03 Still more optimization problems

**Assignment:**

1. A flat piece of metal in the shape of an equilateral triangle with each side having a length of 60 cm is to have material removed from each corner so as to create sides that can be folded up. The seams are to be welded to form a water-tight lidless container. What are the four dimensions of the container if it is to be of maximum volume?

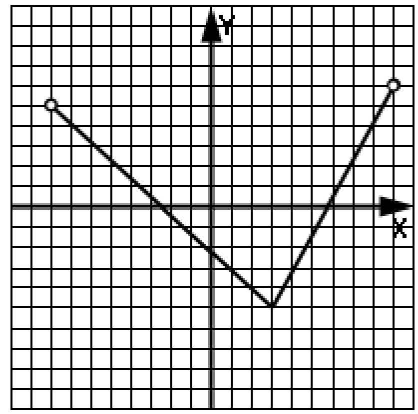
2. A trapezoid is to be formed by starting with a base 2 meters long. The two nonparallel sides both have a length of 5 meters. How far apart should the tops of these sides be in order that the trapezoid is of maximum area?

3. What is the nearest point to the origin on the curve given by $y = 7/x$?

4. A rectangle is arranged so that its top side is on the x-axis. The two corners of the bottom of the rectangle both lie on the parabola, $y = f(x) = (1/4)x^2 - 6$. What are the dimensions of this rectangle if it is to be of maximum area?

# Unit 7:
# Cumulative Review

1. For the function f(x) shown to the right and defined over the open interval (–8, 9), which of the following are true?




A. The 2nd derivative is never negative.

B. The 1st derivative is negative over the interval (–8, 3).

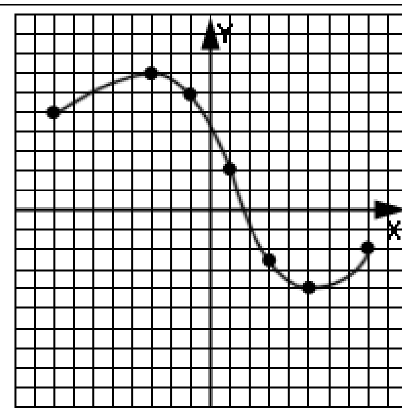
C. The domain of the derivative is (–8, 9).

D. *f* is continuous over (–8, 9).

E. f is everywhere differentiable in (–8, 9).

---

2. The graph of y = f(x) is shown to the right and defined over the interval [–8, 8]. For what value(s) of *x* are both y' and y'' negative?




A. x = –3 B. x = –1 C. x = 1

D. x = 3 E. x = 5

---

3. For the function in problem 2, which of the following are true?

A. Concave up over (–8, 1) B. Concave up over (3, 8)

C. Concave up over (–1, 8) D. Concave down over (–8, 3)

E. None of these

4. $$\lim_{h\to 0}\frac{\sin\left(\frac{\pi}{2}+h\right)-\sin\left(\frac{\pi}{2}\right)}{h}=?$$

A. 1 B. –1 C. 0 D. cos(h) E. Limit doesn't exist.

---

5. A point of inflection for the function f(x) can occur at which one(s) of the following?

A. An *x*-position at which f '' = 0.

B. An *x*-position at which the function changes sign.

C. An *x*-position at which f ' changes sign.

D. An *x*-position at which f '' changes sign.

E. An x-position at which f ' = 0.

---

6. A maximum or minimum for the function f(x) can occur at which one(s) of the following?

A. An *x*-position at which f ' = 0.

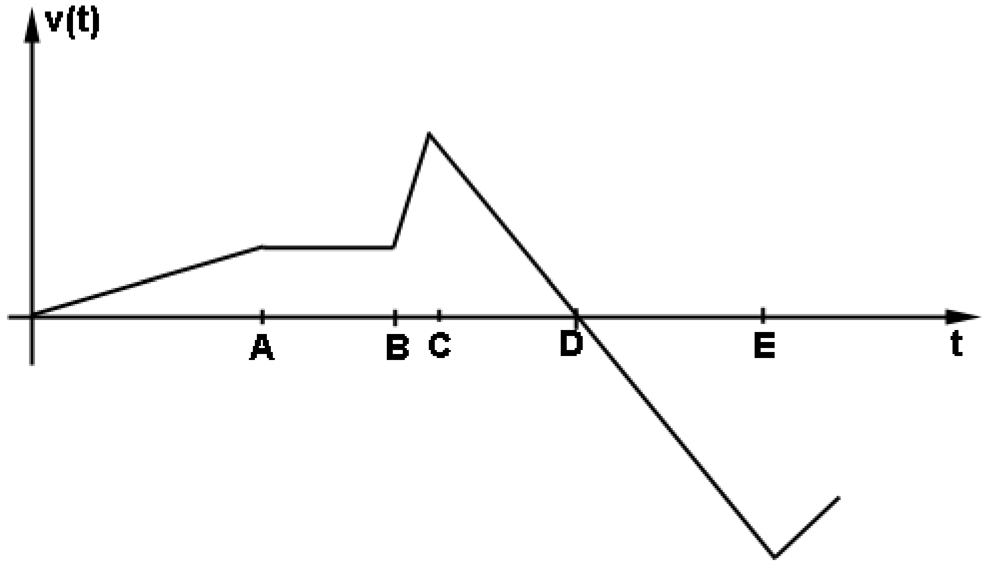
B. An *x*-position at which the function changes sign.

C. An *x*-position at which f ' changes sign.

D. An *x*-position at which f '' changes sign.

E. An x-position at which f '' = 0.

Use the following graph of the velocity of a car as it moves along a straight road to answer problems 7-11.




7. At what time is the car furthest from where it started?

A. A B. B C. C D. D E. E

8. At what time does the car change direction?

A. A B. B C. C D. D E. E

9. At what time is the speed of the car the greatest?

A. A B. B C. C D. D E. E

10. Over what interval is the acceleration of the car the greatest?

A. (A, B) B. (B, C) C. (C, D) D. (D, E)

11. Over what interval is the car going at constant speed?

A. (A, B) B. (B, C) C. (C, D) D. (D, E)

12. For what x-values of $f(x) = 6/(x - 5)$ is the graph concave up?

A. $x > 5$ B. $x < 5$ C. $x > -5$ D. $x < -5$ E. Nowhere

13. What is the equation of the normal line to the curve given by $6 - y^2x^2 = y^3x$ at (2,1)?

14. Consider a 10 ft ladder leaned up against a wall. The bottom of the ladder is pulled along the horizontal floor at a velocity given by $v(t) = 11$ ft/sec. What is the velocity of the top of the ladder when it is exactly 6 ft above the floor?

# Calculus, Unit 8

# Derivatives of inverse, exponential, and logarithm functions

# Unit 8: Lesson 01 Fundamentals of inverse functions and their derivatives

Assuming that **f(x) and g(x) are inverses of each other**, then:

- f( g(x) ) = x
- g( f(x) ) = x
- If (a, b) is an ordered pair of f(x) then (b, a) is an ordered pair of g(x).

To find the inverse of a function, f(x):

- replace *f* with *y*,
- interchange *x* & *y*, and
- solve for *y*. The result is the inverse of f(x)… denoted with $f^{-1}(x)$.

**Example 1:** Find the inverse of f(x) = 2/(x – 5).

$$y = \frac{2}{x-5}$$

reverse

$$x = \frac{2}{y-5}$$

$$xy - 5x = 2$$

$$xy = 5x + 2$$

$$y = \boxed{\frac{5x+2}{x}}$$

---

**Example 2:** Prove that $f(x) = 4x^3$ and $g(x) = \sqrt[3]{\dfrac{x}{4}}$ are inverses.

$$F(g(x)) = 4\left(\sqrt[3]{\frac{x}{4}}\right)^3$$

$$= 4\left(\frac{x}{4}\right)$$

$$= x \checkmark$$

$$g(F(x)) = \sqrt[3]{\frac{\cancel{4}x^3}{\cancel{4}}}$$

$$= \sqrt[3]{x^3}$$

$$= x \checkmark$$

**Example 3:** The functions y = 3x –1 and y = (x + 1)/3 are inverses of each other. Show that the point (5, 14) satisfies the first function and that (14, 5) satisfies the second one. (Notice that these two pairs of coordinates are the reverse of each other.)

$y = 3x - 1$ (5,14)
$14 = 3\cdot5 - 1$
$14 = 15 - 1$
$14 = 14$ ✓

$y = \frac{x+1}{3}$
$5 = \frac{14+1}{3}$
$5 = \frac{15}{3}$
$5 = 5$ ✓

---

Finding the **derivative of the inverse** of a function:

Assume f(x) and g(x) are inverses of each other, so

g ( f(x) ) = x

Take the derivative of both sides with respect to (w.r.t.) *x*:

$g'(f(x))f'(x) = 1$ ← chain rule
$g'(f(x)) = \frac{1}{f'(x)}$
$y = f(x)$
$g'(y) = \frac{1}{f'(x)}$
(x, y) → an ordered pair (a point)
For the function f(x)
This implies the following procedure

**Procedure for finding the derivative of f $^{-1}$(a):**

- Substitute *a* in for *y* in y = f(x).
- Solve for *x*.
- Substitute this value of *x* (call it *b*) into 1/f ′(x) to get 1/f ′(b).

**Example 4:** Find the derivative of $f^{-1}(x)$ at x = 1 where $f(x) = 7x^2 - x + 1$.

$f'(x) = 14x - 1$

$7x^2 - x \cancel{+1} = \cancel{1}$

$7x^2 - x = 0$

$x(7x-1) = 0$

$x = 0\ ;\ x = \frac{1}{7}$

$\frac{d}{dx}\left[f^{-1}\right](1) = \frac{1}{f'(x)}$

$= \frac{1}{14(0)-1} = \boxed{-1}$

or

$= \frac{1}{14(\frac{1}{7})-1} = \frac{1}{2-1} = \boxed{1}$

**Assignment:**

1. Find the inverse of $f(x) = 3x^2 + 1$.

2. Find the inverse of $y = \log_2(x + 5)$.

3. Find any point, (a, b), that satisfies f(x) in problem 1. Then show that (b, a) satisfies the answer obtained in problem 1.

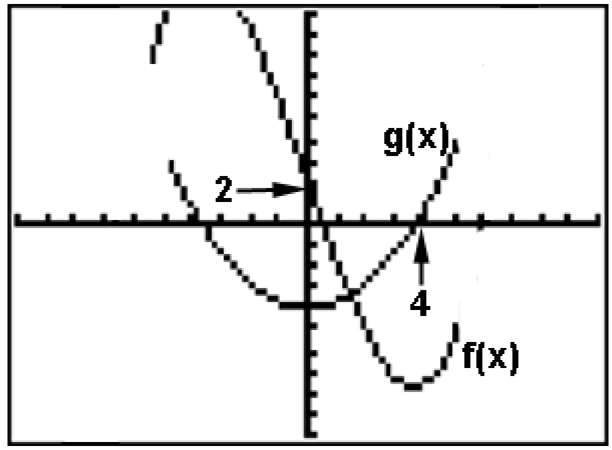
4. Evaluate the derivative of the inverse of f(x) at $x = \sqrt{3}/2$ : $f(x) = \cos(x)$

5. Evaluate $\frac{d}{dx}f^{-1}(x)$ at x = 2 where f(x) = $2x^3 + x - 5$.

6. What is the value of the derivative of the inverse of f(x) = $2\sqrt{x-6}$ when x = 1?

7. Assuming that f(x) and g(x) are inverses of each other, find g'(3) when f(x) = $x^3 - x^2 + 4$.

8. Assuming that g(x) is the derivative of f(x), evaluate the derivative of the inverse of f(x) at x = 2.




The domain of both f(x) and g(x) is (–5, 5).

9. Use the chart below to find the derivative of the inverse of f(x) at x = 2.

| x | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| f(x) | –6 | –3 | 2 | 9 | 18 | 29 | 42 |
| f ’(x) | 2 | 4 | 6 | 8 | 10 | 12 | 14 |

**Unit 8: Lesson 02** **Derivatives of inverse trig functions**

**Derivative rules for inverse trig functions:**

$$\frac{d}{dx}\arcsin(u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\arccos(u) = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\arctan(u) = \frac{1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}\text{arccot}(u) = \frac{-1}{1+u^2}\frac{du}{dx}$$

See **Enrichment Topic I** for how the formulas above are derived.

**Example 1:** Find f ′(x) where f(x) = arcsin($x^3$ – 1).

$$f' = \boxed{\frac{1}{\sqrt{1-(x^3-1)^2}}\,3x^2}$$

**Example 2:** Find f ′(x) where f(x) = $x^2$ arccos($x^2$ + x).

$$f' = \boxed{x^2\frac{-1}{\sqrt{1-(x^2+x)^2}}(2x+1) + \arccos(x^2+x)\,2x}$$

**Example 3:** Find f ′(θ) where f(θ) = arctan(sinθ)

$$f' = \boxed{\frac{1}{1+\sin^2\theta}\cos(\theta)}$$

**Assignment:** In problems 1-7, find the derivative of the given functions.

1. $f(x) = \sin^{-1}(3x)$

2. $f(x) = \cos(\arctan(x)) + x$

3. $f(t) = t^2/\arccos(1 - t)$

4. $f(\theta) = \arcsin(2\theta)\cos(3\theta)$

5. $f(\phi) = 5\arccos^2(\phi^{-1/2})$

6. $f(x) = \arccos(9x) \arcsin(9x)$

---

7. $f(x) = \sqrt[3]{\arccos(x-1)}$

---

8. Evaluate the derivative of $f(x) = \arccos(4x - 6)/3\arctan(x^2)$ at $x = 0$.

9. Find the equation of the tangent line to $f(x) = 11\ \arctan(x^3 + 2)$ at $x = 0$.

---

10. Find the equation of the normal line to $f(x) = \arcsin(4x)$ at $x = .1$.

**Unit 8: Lesson 03** Derivatives of exponential functions

Rule for finding **derivative of a^u** (where *u* is a function of *x*):

$$\frac{d}{dx}a^u = a^u \ln(a)\frac{du}{dx}$$

For a derivation of this rule see **Enrichment Topic G**.

**Example 1:** Find f ’ where f(x) = $5^x$.

$f'(x) = 5^x \ln(5)\, 1$

$= \boxed{\ln(5)\, 5^x}$

**Example 2:** Find f ’ where f(x) = $3^{\sin(2x)}$

$f'(x) = 3^{\sin(2x)} \ln(3) \cos(2x) 2$

$= \boxed{2\cos(2x)\ln(3)\, 3^{\sin(2x)}}$

A special case of the rule above is when a = e:

$$\frac{d}{dx}e^u = e^u \ln(e)\frac{du}{dx}$$

Because ln(e) = 1, the rule becomes:

$$\frac{d}{dx}e^u = e^u \frac{du}{dx}$$

**Example 3:** Find the f ’ where f(x) = sec(x) $e^{\tan(x)}$.

$f' = uv' + vu'$

$f'(x) = \sec(x)\, e^{\tan(x)} \sec^2(x) + e^{\tan(x)} \sec(x)\tan(x)$

$= \boxed{(\sec^3(x) + \sec(x)\tan(x))\, e^{\tan(x)}}$

**Assignment:** In problems 1-10, find the derivative of the given function.

1. $f(x) = e^x$

2. $f(x) = 3e^{2x+1}$

3. $f(x) = 5^{11x}$

4. $f(x) = \sin(4^{4x})$

5. $f(x) = \cos(e^{4x})$

6. $f(x) = \sqrt{e^{x^3}}$

7. $f(x) = e^{4x}/(1 - e^{2x})$

8. $f(x) = x^e + e^x$

9. $f(x) = \ln(6)\ 6^{6x}$

10. $f(x) = \cos(e^x)(x^2 + 7^{ex})$

11. Find the first derivative of *y* w.r.t. *x* where $5 + e^{2xy} - \sin(x) = y^2$.

12. Find the equation of the tangent line at x = 2 to the curve f(x) = $6e^{x^2} - 1310x$.

13. Find $\frac{d^2y}{dx^2}$ where y = cot($e^{x+1}$ + x).

# Unit 8: Lesson 04 Derivatives of logarithm functions

Derivative of a log function:

$$\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{log_a(u)} = \frac{\mathbf{1}}{\mathbf{u}}\mathbf{log_a(e)}\frac{\mathbf{du}}{\mathbf{dx}}$$

See **Enrichment Topic H** for a verification of this derivative rule.

**Example 1:** Find f ’ where f(x) = $\log_3 x$

$$f'(x) = \frac{1}{x}\log_3(e)(1)$$
$$= \boxed{\frac{.91024}{x}}$$

**Example 2:** Find f ’ where f(x) = $\log_2(4x^3)$

$$f'(x) = \frac{1}{4x^3}\log_2(e)\, 12x^2$$
$$= \boxed{\frac{4.328}{x}}$$

A special case of the derivative rule above is when a = e:

$$\frac{d}{dx}\log_e(u) = \frac{1}{u}\log_e(e)\frac{du}{dx}$$

Because $\log_e(u)$ = ln(u) and $\log_e(e)$ = 1, this becomes:

$$\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{ln(u)} = \frac{\mathbf{1}}{\mathbf{u}}\frac{\mathbf{du}}{\mathbf{dx}}$$

**Example 3:** Find f ’ when f(x) = 2x $\ln(x^5)$

$$f' = uv' + vu'$$
$$= 2x\frac{1}{x^5}5x^4 + \ln(x^5)2$$
$$= 10 + \ln(x^5)2 = \boxed{2(5 + \ln(x^5))}$$

Logarithmic differentiation is the process of:

- taking the natural log of both sides of an equation ( y = f(x) ),
- taking the derivative of both sides, and
- then solving for y ’.

This can sometimes make it easier to take the derivative of a function.

**Example 4:** Find y’ when $y = (x^2 + 3)^{1/2}(x^5 - x)^{1/5}$

$$\ln(y) = \ln\left((x^2+3)^{1/2}(x^5-x)^{1/5}\right)$$

$$\ln(y) = \frac{1}{2}\ln(x^2+3) + \frac{1}{5}\ln(x^5-x)$$

$$\frac{1}{y}y' = \frac{1\cdot\cancel{2}x}{\cancel{2}(x^2+3)} + \frac{5x^4-1}{5(x^5-x)}$$

$$y' = y\left[\frac{x}{x^2+3} + \frac{5x^4-1}{5(x^5-x)}\right]$$

$$y' = \boxed{(x^2+3)^{1/2}(x^5-x)^{1/5}\left[\frac{x}{x^2+3} + \frac{5x^4-1}{5(x^5-x)}\right]}$$

**Assignment:** In problems 1-10, find the derivative of the given functions.

1. $f(x) = \log_4(x)$

2. $f(x) = 2\log_5(6x^4)$

3. $g(x) = \ln(\cos 7x)$

4. $k(x) = \log(x^3 + e^x)$

5. $f(x) = 20x^2\ln(x + 1)$

6. $f(x) = \ln(x)/(2x^3)$

7. $f(x) = \ln|x + 1|$

8. $f(x) = \ln( (x - 4)/\sqrt{x+6} )$

9. $q(x) = f( g(x) )$ where $f(x) = x^2$ and $g(x) = \log_3(x)$.

10. $h(x) = \ln( \ln(\sec(x)) )$

11. Use logarithmic differentiation to find the derivative of:
$y = x(1 + x^2)^2/\sqrt{3x - 2}$

12. Use implicit differentiation to find y' from $y^2e^{2x+1} + \ln(y) + x^2 - x = 0$.

13. Find the second derivative of *y* w.r.t. *x* where $y = 10x \ln(x) + e^{4x}$.

# Unit 8: Cumulative Review

1. Involved in finding the absolute maximum of the function f(x) is:

   A. locating the roots of the function.

   B. finding the x-positions at which f '' changes sign.

   C. finding the x-positions at which f ' changes sign.

   D. examining the end points of the interval over which f(x) is defined.

   E. more than one of these

---

2. $$\lim_{\theta\to 0}\frac{\sin^2\theta}{3(1-\cos\theta)} = ?$$

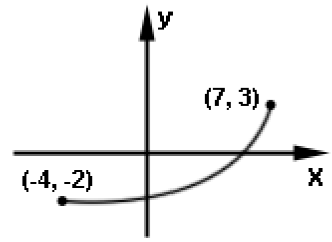
   A. 2/3  B. 1/3  C. 0  D. sinθ  E. Doesn't exist

---

3. $$\lim_{x\to\infty}\frac{5x^3+x}{2\pi x^3-x^2+x} =?$$

   A. 0  B. 5/2  C. 5/(2π)  D. +∞  E. –∞

4. The graph of y = f(x) is shown here on the closed interval [–4, 7]. As can be seen, this function is continuous and differentiable everywhere on (–4, 7). There exists an x = c value, $-4 \le c \le 7$, such that:




A. f ’(c) = 5/11 B. f ’(c) = 0 C. f(c) = 5/11

D. f(c) = 0 E. f ’(c) = 11/5 F. More than one of these

---

5. The function $f(x) = -x^3 + 1x + 6$ is

A. decreasing everywhere.
B. increasing everywhere.
C. decreasing over the interval $(-\sqrt{1/3}, \sqrt{1/3})$.
D. increasing over the interval $(-\sqrt{1/3}, \sqrt{1/3})$.
E. decreasing over the interval $(-\sqrt{1/3}, 0)$ and increasing over the interval $(0, \sqrt{1/3})$.

6. The function $f(x) = 5x^3 - 2kx^2$ has a point of inflection at $x = 4$ for what value of k?

A. –30 B. 30 C. –15 D. 15 E. 15/4

---

7. If $f(8) = 11$, $f'(8) = 0$, and $f''(8) = -3$ then the point (8, 11) is what kind of point on the graph of f(x)?

A. a relative maximum B. a relative minimum C. a root

D. a point of inflection E. a critical point F. more than one of these

---

8. If $h(x) = (x^3 + x)/g(x)$, $g(2) = 5$, and $g'(2) = -1$ then $h'(2) = ?$

A. –3 B. 3 C. –11/5 D. 11/5 E. none of these

9. Assuming $y = \cos(u)$, *u* is a function of *x*, and *u*' represents the derivative of *u* w.r.t *x*, then the derivative of *y* w.r.t. *x* is represented as

A. $-\sin(u)$ B. $-\sin(u)\,u'$ C. $-\sin(u')$ D. $\cos(u)\,u' + u(-\sin(u))$

E. $\sin(u)\,u'$ F. $\cos^2(u)/2$

---

10. Consider an object moving along the line $y = 2x$ such that its projection on the x-axis is moving at 3 cm/sec. If an observer at (4, 0) is watching the object through a telescope, at what rate will the telescope be rotating when the object's x-position is 8?

# Unit 8: Review

1. Use the table below to find the derivative of the inverse of f(x) evaluated at x = –3.

| x | –3 | –1 | 1 | 3 | 5 | 7 | 9 | 11 | 13 |
|---|---|---|---|---|---|---|---|---|---|
| f(x) | –9 | –7 | –5 | –3 | – 1 | –1 | 3 | 5 | 7 |
| f'(x) | –12 | –3 | 0 | 11 | 15 | 8 | 4 | 3 | 1 |

2. g(x) is the inverse of $f(x) = 4x^2 + x - 1$. What is the derivative of g(x) evaluated at x = 11?

3. Find g'(x) where g(x) = arctan(2x – sin(x)).

4. What is the derivative of $f(x) = \sqrt{e^{4x} + x}$ ?

5. Find $h'(x)$ where $h(x) = f(x)\, g(x)$, $f(x) = \log_2(x^2)$, $g(x) = \arccos(x + 2)$.

---

6. What is $y'$ where $y = e^x/\ln(\cot(x))$?

---

7. Find $f\,'(x)$ where $f(x) = \arcsin(\log_4 x)$.

---

8. Find $f\,'(x)$ where $f(x) = \ln(\log_{15} e^x)$

9. Find $f'(x)$ where $f(x) = \arcsin(3x/5)$. Leave the answer in a form in which there are no fractions under a radical and no radicals in a denominator.

---

10. Find $f''(x)$ where $f(x) = \ln(x^5 - x^4)$.

---

11. Find $f''(x)$ where $f(x) = 9a^{2x+1}$.

12. Find y’ where $x \ln(y) + x e^{y} = x$

13. What is the equation of the tangent line to the graph of $f(x) = \arctan(3x)$ at $x = 1$?

# Calculus, Unit 9

# Antiderivatives
# (Indefinite Integrals)

# Unit 9: Lesson 01 Basic integration rules, integrating polynomials

Consider taking the derivative of a single term of a polynomial:

$f(x) = 5x^3$

$f'(x) = 15x^2$

We did this by *multiplying by the power* (3) and then **subtracting 1 from the power**.

Working **backwards** on this process is known as **integration** or taking the **antiderivative**.

Specifically, for the problem above, we would begin with the answer ($15x^2$),

- raise the power by 1 and
- then divide by the new power.

$$\frac{15x^3}{3} = 5x^3$$

This is almost right. What if the original problem had been to take the derivative of $f(x) = 5x^3 + 7$ or perhaps $f(x) = 5x^3 + 10$? In all cases the derivative would have still been $f'(x) = 15x^2$.

To account for the constant that could clearly be anything, here are the modified rules for integrating a term of a polynomial:

- raise the power by 1,
- divide by the new power, and
- add an arbitrary constant (call it C)

The **symbolism for integrating** a function, f(x), is $\int \mathbf{f(x)dx}$.

**Example 1:** $\int 6x^2 dx$ = ?

$$\int 6x^2 dx = \frac{6x^3}{3} + C$$
$$= \boxed{2x^3 + C}$$

**Example 2:** $\int -11x^6 dx$ = ?

$$\int -11x^6 dx = \boxed{-\frac{11x^7}{7} + C}$$

**Integration rule summary:**

- $\int 0dx = C$ (Recall that the derivative of a constant is 0.)
- $\int dx = x + C$ (Think of the function as f(x) = $x^0$.)
- $\int k\, f(x)dx = k \int f(x)dx$ (*k* is a constant)
- $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$
- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ where n ≠ −1
  (This even works when *n* is negative or fractional.)

**Example3:** $\int (4x^5 - 12x + 1)\, dx$ = ?

$$\int (4x^5 - 12x^1 + x^0)dx$$
$$= \frac{4x^6}{6} - \frac{12x^2}{2} + \frac{x^1}{1} + C$$
$$= \boxed{\frac{2}{3}x^6 - 6x^2 + x + C}$$

**Example 4:** $\int 4x^{-2} dx$ = ?

$$4\int x^{-2} dx$$
$$= \frac{4x^{-1}}{-1} + C$$
$$= \boxed{\frac{-4}{x} + C}$$

**Example 5:** $\int \left(7\sqrt{x} + \frac{1}{x^2}\right) dx$ =?

$$\int (7x^{1/2} + x^{-2})dx = \frac{7x^{3/2}}{3/2} + \frac{x^{-1}}{-1} + C$$
$$= \boxed{\frac{14}{3}x^{3/2} - \frac{1}{x} + C}$$

**Example 6:** $\int \dfrac{(x+2)^2}{\sqrt{x}}dx = ?$

$$\int (x^2+4x+4)x^{-1/2}dx$$
$$= \int (x^{3/2}+4x^{1/2}+4x^{-1/2})dx$$
$$= \frac{x^{5/2}}{5/2} + \frac{4x^{3/2}}{3/2} + \frac{4x^{1/2}}{1/2} + C$$
$$= \boxed{\frac{2x^{5/2}}{5} + \frac{8x^{3/2}}{3} + 8x^{1/2} + C}$$

The integrals introduced in this lesson are known as **indefinite integrals** and are denoted with just a plain integral sign … $\int f(x)dx$ .

**Definite integrals** that we will learn about later, have an upper and lower limit and are denoted like this… $\int_2^{11} f(x)dx$ .

**Assignment:** Integrate (find the antiderivative of) the functions in problems 1-8:

1. $f(x) = x$
2. $f(x) = 5$
3. $g(x) = 7x^5 - 36x^2 + x$
4. $y = 1/x^2$
5. $f(x) = \sqrt[3]{x}$
6. $f(x) = (x + 5)^2$
7. $g(x) = (\sqrt{x} + 8)/x^3$
8. $f(x) = (5/6)x^4 - 8x^3 + x^2 - 3x + 9$

9. Find the antiderivative of $1/(x^2\sqrt{x})$.

10. Find the antiderivarive of f(x) = 0.

11. $\int (6y^{\frac{3}{5}} - 11\sqrt[3]{y})\, dy = ?$

12. $\int dt + 2\int (13t)\, dt$

13. $\int x(x^3 - 4)^2 dx$

14. $\int 10(x^2 + x^1 + x^{-1} + x^{-2})dx$

15. Give $f'(x) = 5x^{3/2} - 6x + 2$ find f(x).

16. Given $f'(x) = x^3/(x^2 + 11)^{-2}$ find f(x).

# Unit 9: Lesson 02 More integration practice

**Example 1:** Find the indefinite integral of f(x) = $(\sqrt[3]{x^7} + x)\sqrt{x}$ .

$$\int (x^{7/3} + x)x^{1/2}dx$$

$$= \int (x^{17/6} + x^{3/2})dx = \frac{x^{23/6}}{23/6} + \frac{x^{5/2}}{5/2} + C$$

$$= \boxed{\frac{6x^{23/6}}{23} + \frac{2x^{5/2}}{5} + C}$$

---

**Example 2:** If f ’(x) = ( 1/(x$\sqrt{x}$) )$^3$ find f(x).

$$\int \frac{1}{(x \cdot x^{1/2})^3} dx = \int \frac{1}{(x^{3/2})^3} dx$$

$$= \int x^{-9/2} dx = \frac{x^{-7/2}}{-7/2} + C$$

$$= \boxed{\frac{-2}{7x^{7/2}} + C}$$

**Assignment:** In problems 1-7, find the indefinite integral of the given function.

1. $f(z) = az^3$ where *a* is a constant

2. $g(p) = p^3 + p^2 + 1$

3. $f(x) = 5x^K + 4 + k$ where *k* is a constant

4. $f(x) = 97$

5. $f(x) = (x - 1)^3$

6. $h(x) = 1/(x^{5/3} - 6x + 14)^{-1}$

7. $f(x) = (3x + 7)(2x + 1)(x + 1)^2$

In problems 8-10, the derivative of a function is given. Find the original function.

8. $f'(x) = (\sqrt[3]{x} + x)^2$

9. $f'(x) = \sqrt{x^2 - 14x + 49} + 4x$

10. $g'(x) = a\sqrt[b]{x}$ where *a* and *b* are constants

In problems 10-14, find the antiderivatives of the given functions:

| | |
|---|---|
| 11. f(x) = sin(a) + ax where *a* is a constant | 12. f(x) = 3π |
| 13. $h(x) = 11x^{100} + x^{-9} + 1$ | 14. $m(x) = 1/x^p$ (What is a forbidden value of *p* when taking the antiderivative?) |

# Unit 9: Lesson 03 Integrating trig functions

**Rules for integrating trig functions:**

$$\int \sin(x)\,dx = -\cos(x) + C \qquad \int \sec(x)\tan(x)\,dx = \sec(x) + C$$

$$\int \cos(x)\,dx = \sin(x) + C \qquad \int \csc(x)\cot(x)\,dx = -\csc(x) + C$$

$$\int \sec^2(x)\,dx = \tan(x) + C \qquad \int \csc^2(x)\,dx = -\cot(x) + C$$

**Example 1:** $\int (\, 2\sin(x) + \cos(x)\,)\ dx = ?$

$$= 2\int \sin(x)\,dx + \int \cos(x)\,dx$$

$$= \boxed{-2\cos(x) + \sin(x) + C}$$

**Example 2:** $\int \csc(x)\,(\,\cot(x) - \csc(x)\,)\ dx = ?$

$$= \int (\csc(x)\cot(x) - \csc^2(x))\,dx$$

$$= \int \csc(x)\cot(x)\,dx - \int \csc^2(x)\,dx$$

$$= \boxed{-\csc(x) + \cot(x) + C}$$

**Assignment:**

1. $\int 3\sin(x)\,dx = ?$

2. $4\int \sec(\theta)\tan(\theta)\,d\theta = ?$

3. $\int(\csc^2(\theta) + \cos(\theta))d\theta = ?$

4. $\int \sec^2(\theta)d\theta = ?$

5. $\int (\sec(z)[\sec(z) + \tan(z)])\,dz = ?$

6. $\int(1 + \tan^2(y))\,dy = ?$

7. $\int (7\sqrt{x} + 4\cos(x))\ dx = ?$

8. $2\int (\sin(x) - \cos(x))\ dx = ?$

9. $\int \sqrt{1 - \sin^2(x)}\ dx = ?$

10. $\int \sqrt{\pi \sin^2 x}\ dx = ?$

11. $\int (1 + \cot^2 x)\ dx = ?$

12. $(1/3)\int (2\sin(x) + \sec^2 x)\ dx = ?$

13. If $f'(x) = 36\sin(x) + \csc^2(x)$, what is $f(x)$?

---

14. What is $g(p)$ if its derivative is $11(\cos^2(p) + \sin^2(p))\cos(p)$?

---

15. Find $f(x)$ if $f'(x) = \sin(x)/(\cos^2(x))$.

# Unit 9: Lesson 04 Integration using the chain rule in reverse

Consider using the chain rule in taking the derivative of f(x) = sin(3x).

**Example 1:** Find f ′(x) where f(x) = sin(3x).

$$f'(x) = \boxed{3\cos(3x)}$$

Now, let's think backwards on this process and find $\int 3\cos(3x)\,dx$. This is easy because we can "cheat" and look at where we started in Example 1 above for the answer.

**Example 2:** $\int 3\cos(3x)dx = ?$

Because of the chain rule, the derivative of "what's inside" must be sitting here beside it.

$$\int 3\cos(3x)\,dx = \boxed{\sin(3x) + C}$$

**Example 3:** $\int 4\sqrt{4x+2}\,dx = ?$

Because of the chain rule, the derivative of "what's inside" must be sitting here beside it.

$$\int 4(4x+2)^{1/2}dx = \frac{(4x+2)^{3/2}}{3/2} + C = \boxed{\frac{2(4x+2)^{3/2}}{3} + C}$$

**Example 4:** $\int 22x\sqrt[3]{11x^2-5}\,dx = ?$

Because of the chain rule, the derivative of "what's inside" must be sitting here beside it.

$$\int 22x(11x^2-5)^{1/3}dx = \frac{(11x^2-5)^{4/3}}{4/3} + C = \boxed{\frac{3(11x^2-5)^{4/3}}{4} + C}$$

Consider $\int \sin(4x)\,dx$ where the "derivative of "what's inside" (4) is not present as it needs to be in order to work backwards on the chain rule.

Simply multiply by 1 inside the integral in the form of (1/4) (4) and let the 1/4 migrate through the integral sign (as it can since it's a constant).

---

**Example 5:** $\int \sin(4x)\,dx = ?$

$$\int \frac{4}{4} \sin(4x)\,dx = \frac{1}{4}\int 4\sin(4x)\,dx$$

$$= \boxed{-\frac{1}{4}\cos(4x) + C}$$

**Warning!**

Consider the integral, $\int \sin(4x^2)\,dx$, where the derivative of "what's inside" is 8x. Could we do the following in an attempt to work this?

$$\int \frac{8x}{8x}\sin(4x^2)\,dx = \frac{1}{8x}\int 8x\sin(4x^2)dx \;???$$

**No!** The *x* in 1/(8x) in front of the integral sign is illegal. **Variables cannot migrate across the integral sign** as did the constant 8 in this case.

With more advanced techniques, it is possible to integrate $\sin(4x^2)$ but for us, it can't be done with the limited tools that we have developed at this point in our study of calculus.

**Assignment:** If an integral can't be done with the limited techniques we have developed so far, then stat, "can't be done."

1. $\int 9\cos(9x)\ dx$

2. $\int 2\sec(2x)\ \tan(2x)\ dx$

3. $\int \cos(x)\sqrt{\sin(x)}\ dx$

4. $\int \sqrt{17x}\ dx$

5. $\int \cos(-2x)\ dx$

6. $\int \cos(5x)\sqrt{\sin(5x)}\ dx$

7. $\int \cos(2\pi x/3)\ dx$

8. $\int \sec^2(\theta/5)\ d\theta$

9. $\int \csc(8x+1)\cot(8x+1)\,(\csc(8x+1))^2\ dx$

10. $\displaystyle\int \frac{x+1}{(4x^2+8x)^2}\,dx$

11. $\displaystyle\int (3x+4)\,(x^3+4x)^5\ dx$

12. $\int (\,1 + \tan^2(2x)\,)\,dx$

13. $\int \sqrt{x^2 - 9}\,dx$

14. $\int \frac{\tan(x)}{\cos(x)}\,dx$

**Unit 9: Lesson 05**

# Evaluation of integration constants
# An application of integration

Given the derivative f ′(x) of a function and a point on the original function f(x), it is possible to **evaluate the constant** produced by an indefinite integral:

- Integrate f ′(x) to produce f(x) …it includes a constant *C*.
- Substitute the point into the resulting equation and evaluate *C*.

**Example 1:** Find f(x) when f ′(x) = sec$^2$(x) and f(π/6) = 12.

$f(x) = \int \sec^2(x)\,dx = \tan(x) + C$

$12 = \tan(\pi/6) + C$

$12 = \frac{1}{\sqrt{3}} + C$

$C = 12 - \frac{\sqrt{3}}{3}$

$\boxed{F(x) = \tan(x) + 12 - \frac{\sqrt{3}}{3}}$

Given the second derivative of a function f ′′(x) and two points, it is possible to **evaluate the constants** produced in integrating back to f(x):

(One point can be on f ′(x) and the other on f(x), or both on f(x).)

- Integrate f ′′(x) to produce f ′(x) which includes a constant C1.
- If one of the given points is on f ′(x), substitute it in and evaluate C1 to produce a complete f ′(x).
- Integrate f ′(x) to produce f(x) … it includes a constant C2.
  - If one of the given points was on f ′(x), then substitute in the remaining point and evaluate C2 to produce a complete f(x).
  - If both points are on f(x), substitute in one at a time to produce two linear equations. Solve them for C1 and C2 so as to produce a complete f(x).

**Example 2:** Find f(x) where f ''(x) = $x^3 - x^2$, f'(2) = –1, and f(1) = 3.

$$f'(x) = \int (x^3 - x^2)\,dx = \frac{x^4}{4} - \frac{x^3}{3} + C$$
$$f'(2) = -1 = \frac{2^4}{4} - \frac{2^3}{3} + C1$$
$$C1 = -1 - 4 + \frac{8}{3} = -2.\overline{3} \rightarrow f'(x) = \frac{x^4}{4} - \frac{x^3}{3} - 2.\overline{3}$$
$$f(x) = \int \left(\frac{x^4}{4} - \frac{x^3}{3} - 2.\overline{3}\right)dx$$
$$= \frac{x^5}{20} - \frac{x^4}{12} - 2.\overline{3}x + C2$$
$$f(1) = 3 = \frac{1}{20} - \frac{1}{12} - 2.\overline{3} + C2$$
$$C2 = 3 - \frac{1}{20} + \frac{1}{12} + 2.\overline{3} = 5.3\overline{6}$$
$$f(x) = \boxed{\frac{x^5}{20} - \frac{x^4}{12} - 2.\overline{3}x + 5.3\overline{6}}$$

**Example 3:** Find f(x) where f ''(x) = $x^2 - x$, f(1) = –3, and f(0) = 4.

$$f'(x) = \int (x^2 - x)\,dx = \frac{x^3}{3} - \frac{x^2}{2} + C1$$
$$f(x) = \int \left(\frac{x^3}{3} - \frac{x^2}{2} + C1\right)dx = \frac{x^4}{12} - \frac{x^3}{6} + C1x + C2$$
$$f(1) = -3 = \frac{1}{12} - \frac{1}{6} + C1 + C2 \rightarrow C1 + C2 = -2.91\overline{6}$$
$$f(0) = 4 = 0 - 0 + C1\cdot 0 + C2 \rightarrow C2 = 4$$
$$C1 + C2 = -2.91\overline{6}$$
$$C1 + 4 = -2.91\overline{6}$$
$$C1 = -6.91\overline{6}$$
$$f(x) = \boxed{\frac{x^4}{12} - \frac{x^3}{6} - 6.91\overline{6}x + 4}$$

In physics when given a position function of time s(t), the velocity function v(t) is produced by taking the derivative. The acceleration function a(t) is produced by taking the derivative of the velocity.

Likewise, beginning with the acceleration function, integrate twice to produce v(t), and s(t) respectively.

$$s(t) \xrightarrow{\text{derivative}} s'(t) = v(t) \xrightarrow{\text{derivative}} s''(t) = v'(t) = a(t)$$

$$s(t) \xleftarrow[\text{integrate}]{} s'(t) = v(t) \xleftarrow[\text{integrate}]{} s''(t) = v'(t) = a(t)$$

**Example 4:** An object is moving along a straight line with a velocity given by v(t) = $3t^2 - 6t + 1$. Find both the acceleration and position functions when the object is at s = 5 when t = 2.

$a(t) = v'(t) = \boxed{6t-6}$

$s(t) = \int(3t^2-6t+1)dt = \frac{3t^3}{3} - \frac{6t^2}{2} + t + C$

$s(t) = t^3 - 3t^2 + t + C$

$5 = 2^3 - 3\cdot 2^2 + 2 + C$

$5 - 8 + 12 - 2 = C$

$C = 7$

$s(t) = \boxed{t^3 - 3t^2 + t + 7}$

**Assignment:**

1. Find f(x) when $f'(x) = \sin(2x)$ and $f(0) = 4$.

2. Find g(x) when it passes through (4, –1) and $g'(x) = x + 2$.

3. The position of an object moving in a straight line is at position 0 at $t = 2$ and has a velocity given by $v(t) = 3t - 4t^2$. Find both the position and acceleration functions of time.

4. Miss Coral J., an avid UFO buff, reports sighting a UFO traveling in a straight line with an acceleration function $a(t) = 3t + 1$. During the sighting she observed that the position was 10 and the velocity was $-1$ when t was 0. In preparing her report for the authorities, she needs both the velocity and position as functions of time. As her able assistant, you are to provide those two functions.

---

5. Find $f(x)$ given that the points (0, 0) and (1, $-1$) lie on the graph of the function and that $f''(x) = 4x + 12$.

# Unit 9: Lesson 06 Indefinite integrals with a graphing calculator

The commonly used TI 84 graphing calculator is **not** capable of being given a function and then giving back the indefinite integral function.

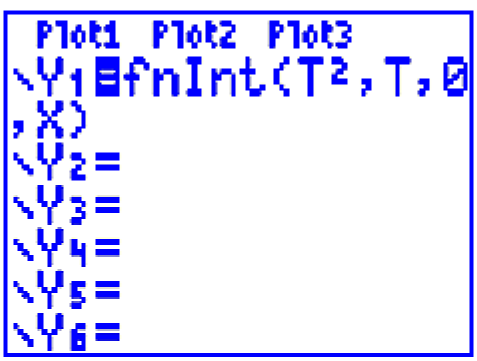
It is, however, capable of **producing the graph** of an indefinite integral.

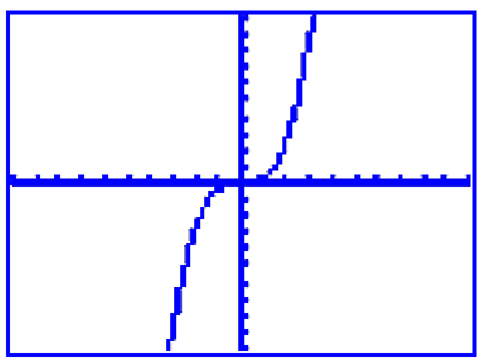
Consider the indefinite integral $\int x^2dx = \dfrac{x^3}{3} + C$.

> The function $f(x) = x^3/3 + C$ produces a **family of curves** as *C* takes on various values.

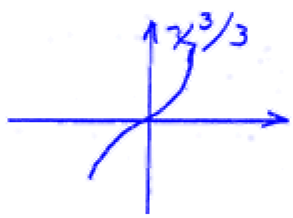
See **Calculator Appendix AG** for how to produce the graph of an indefinite integral on a graphing calculator. The graph produced will be of correct general shape **except for its vertical shift**. That depends on the constant of integration as determined by one of the parameters entered into the calculator.

**Example 1:** Produce the graph of the indefinite integral of $f(x) = x^2$ on the graphing calculator. Compare this graph to the graph of the integral of $f(x) = x^2$ produced by hand. (Assume the constant of integration is 0.)



$\int x^2dx = \boxed{\dfrac{x^3}{3} + C}$



**Assignment:**

1. Produce the graph of the indefinite integral of $f(x) = x$ on the graphing calculator. Compare this graph to the graph of the integral of $f(x) = x$ produced by hand. (Assume the constant of integration is 0.)

---

2. Produce the graph of the indefinite integral of $f(x) = \cos(x)$ on the graphing calculator. Compare this graph to the graph of the integral of $f(x) = \cos(x)$ produced by hand.

Proceed to the review for Unit 9.

# Unit 9: Review

1. $\int (x^4 + x)\,dx = ?$

2. $\int \frac{(x-5)^2}{\sqrt[3]{x}}\,dx = ?$

3. $\int \sqrt{x}\left(x^{1/3} - 2x^{4/5}\right)\,dx = ?$

4. $\int (x^n - 2)^2 dx = ?$ (*n* is a constant)

5. $\int \left(\frac{2x^{-6}}{3} + \frac{1}{x^2}\right)$dx = ?

6. $\int 4\sqrt[r]{x^k}\,dx = ?$ (*k* and *r* are integer constants)

7. If $f'(\theta) = 3\cos(\theta)$ find $f(\theta)$.

8. If $f'(m) = 1 + \cot^2(m)$ find $f(m)$.

9. Find the antiderivative of $\cot(2x)/\sin(2x)$

10. Find the antiderivative of $\sqrt{x} + \sec^2(3x)$

11. $\int \frac{\sqrt{2x^2+1}}{x^{-1}}\,dx = ?$

12. $\int \frac{\tan(2\theta)}{\cos(2\theta)}\,d\theta = ?$

13. Find $f(x)$ when $f'(x) = \csc^2 x$ and $f(\pi/2) = 4$.

---

14. Find $f(x)$ when $f''(x) = x + \cos(3x)$, $f'(\pi/3) = 0$, and $f(0) = 1$.

15. Find f(x) when $f''(x) = 1/\sqrt{2x+1}$, f(0) = 4, and f(1) = 2.

16. If the velocity of an object is given by $v(t) = t^2 + 7t^3$ and its position at t = 1 is 11, find acceleration a(t) and position s(t) as functions of time.

17. Assuming that the lower limit parameter is 0, use a graphing calculator to make a sketch of $\int x\cos(3x)\,dx$. Set the window to x → [–2, 2], y → [–1, 1].

**Sem 1: Review**

# Comprehensive Review

1. $\lim\limits_{x \to -8} \dfrac{x^2 + 3x - 40}{x + 8} = ?$

---

2. $\lim\limits_{x \to \infty} \dfrac{4x^3 - 6x}{2x^3 + x^2} = ?$

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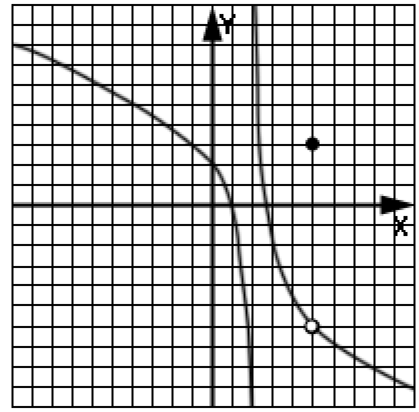
3. $\lim\limits_{x \to 5} \dfrac{\sqrt{5} - \sqrt{x}}{x - 5} = ?$

4. Find the following for f(x) = y




$\lim_{x\to 2^-} f(x) =$

$\lim_{x\to 2^+} f(x) =$

$\lim_{x\to 5} f(x) =$

$f(2) =$

$f(5) =$

---

5. Determine the value of D so as to insure that this function is everywhere continuous.

$$f(x) = \begin{cases} 2Dx^3 & \text{if } x \leq 2 \\ -5 & \text{if } x > 2 \end{cases}$$

---

6. Sketch this piecewise function and then answer the questions.

$$f(x) = \begin{cases} x & when\ x < -3 \\ 2 & when\ x = -3 \\ -\sqrt{x+3} - 1 & when\ x > -3 \end{cases}$$

$\lim_{x\to -3^-} f(x) =$

$\lim_{x\to -3^+} f(x) =$

$\lim_{x\to -3} f(x) =$

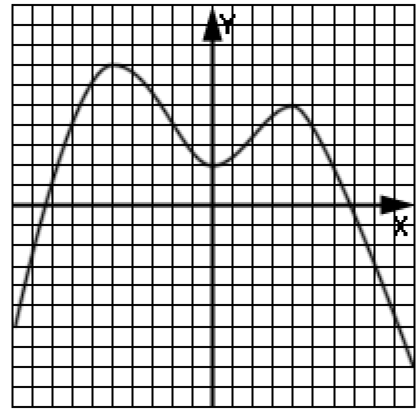
$f(-3) =$

7. Find the instantaneous rate of change of $f(x) = 3x^2 - 4x$ at $x = 1$.

8. Using the formal definition of the derivative, find the derivative of $f(x) = \sqrt{x+1} - 2$ at $x = 1$.

9. Using the formal definition of the derivative, find the slope of the normal line to the curve given by $f(x) = 1/(x - 6)$ at $x = -3$.

10. Using the function whose graph is shown to the right, specify the following intervals:




a. Interval(s) of negative derivative

b. Interval(s) of positive slope

c. Interval(s) of decrease

---

11. Determine by an analysis of a continuity test and "left" & "right" derivatives if this function is differentiable at x = –2.

$$f(x) = \begin{cases} 5x^2 & \text{if } x \le -2 \\ -20x - 20 & \text{if } x > -2 \end{cases}$$

---

12. Which of the following is a correct representation of the derivative of f(x) evaluated at *a*?

A. $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$

B. $\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$

C. $\lim\limits_{x\to a}\dfrac{f(x+h)-f(x)}{h}$

D. More than one of these

13. Prove that the derivative of sec(x) is sec(x) tan(x).

14. Find f ′(θ) when f(θ) = sin(θ) cot(θ).

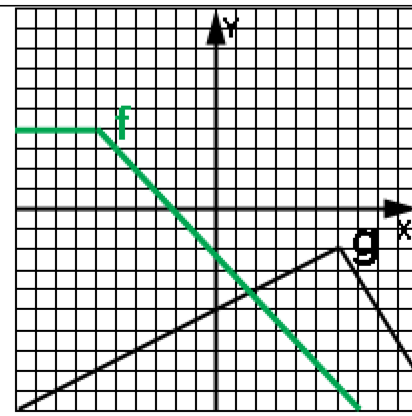
15. Find g′(t) when g(t) = $3t^2/(t-5)$.

16. Find the equation of the tangent line (at x = 2) to the curve given by
$f(x) = (x^2 + 7)\left(\sqrt{x}\right)^3$.

17. Find the derivative of the following piecewise function:

$$f(x) = \begin{cases} -x^4 + 2x & \text{if } x < -5 \\ x^3 & \text{if } x \geq -5 \end{cases}$$

18. Using the functions f(x) and g(x) from the adjacent graph, find p'(–4) where p(x) = f(x) g(x).




19. Find h'(x) given h(x) = $8x^3\sqrt{x^4 - 7}$ .

20. Find f '(x) where f(x) = cos( sec(x) ).

21. Find $\frac{d^3y}{dx^3}$ when y = 5 sin(6x).

22. Find the equation of the normal line to f(x) = $\sqrt{\tan(x)}$ at x = π/4 radians.

---

23. An object is moving rectilinearly in time according to s(t) = $3t^2 - 6t + 1$ meters over the time interval [0, 4] seconds. Find the velocity and acceleration as functions of time and give the appropriate units of each. Evaluate the s(t), v(t), and a(t) functions at t = 1 sec.

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24. Differentiate implicitly to find y' where $x^3y = x\sin(y)$.

25. Find the x-position(s) at which the normal lines to the curve given by $xy^2 = 9$ are parallel to $4x + 8y=1$.

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26. Assume $f(0) = 2$, $f'(0) = -4$, $f''(0) = 3$, and $g(x) = \tan(4x)\, f(x)$. What is the equation of the line tangent to $g(x)$ at $x = 0$?

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27. Use the data in this chart to find $g'(2)$ where $g(x) = f(2x)$.

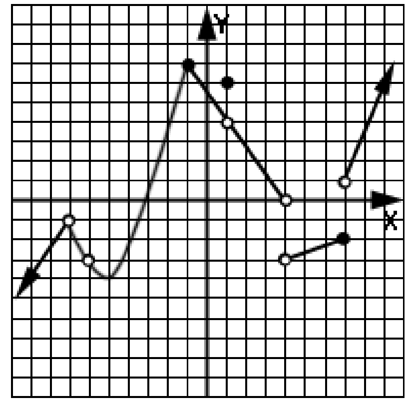
| | | | | | | |
|---|---|---|---|---|---|---|
| x | 1 | 2 | 3 | 4 | 5 | 6 |
| f '(x) | .2 | .3 | .5 | .7 | .9 | 1 |

28. A huge spherical snowball is melting at the rate of 50 $cm^3$/sec. What is the rate of change of the radius when the radius is 2 meters?

29. Consider the function $f(x) = x^2 + 4x – 77$ over the interval [–11, 7]. Show that the conditions of Rolle's Theorem are met over this interval and then find the value of ξ predicted by the theorem.

30. Consider the function f(x) = 4sin(x) + x over the interval [0, π]radians. Show that the conditions of the Mean Value Theorem are met over this interval and then find the value of ξ predicted by the theorem.

31. For each *x* value below, state if it is a critical value or not. In each case justify your answer.




A. x = –7

B. x = –6

C. x = –5

D. x = –1

E. x = 1

F. x = 4

G. x = 7

32. Find the intervals of concavity and points of inflection of $f(x) = .25x^4 - x^3 - 12x$.

33. A rectangular sheet of metal 10 in X 20 in will have small squares cut out of each corner and then discarded. The four “sides” are then folded up so as to form a lidless rectangular box. What size squares should be cut out of each corner so as to maximize the volume of the box? What is the resulting maximum volume?

34. The strength of a rectangular wooden beam is directly proportional to the width of the beam’s cross section and to the square of its depth. What are the dimensions of a rectangular beam that could be cut from a rectangular log of radius R?

35. g(x) is the inverse of f(x) = $x^2 - 6x - 20$. What is derivative of g(x) evaluated at x = 20?

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36. Find p'(x) where p(x) = arctan( 4x – cos(x) )

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37. What is the derivative of f(x) = $\sqrt[3]{e^{3x} + x}$ ?

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38. Find f '(x) where f(x) = arccos($3x^2/5$). Leave the answer in a form in which there are no fractions under a radical and no radicals in a denominator.

39. $\int \frac{(t-4)^2}{\sqrt[3]{t}}\, dt = ?$

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40. $\int (x^{3n} - 1)^2 dx = ?$ (*n* is a constant)

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41. If $f'(x) = (2x^2 + 1)/x^{-1}$, find f(x).

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42. What is f(x) when $f'(x) = \sec^2(8x)$ and $f(2\pi/3) = 2$?

43. Find f(x) when $f''(x) = x + \sin(5x)$, $f'(\pi/6) = 0$, and $f(0) = 1$.

44. Find f ’(x) where $f(x) = (\log_3(x^2 + 1))^4 + x$