Blue Pelican Pre-Calculus

First Semester

Absent-student Version 1.01

Pre Calculus Syllabus (First Semester)

Unit 1: Algebra review

Lesson 01: Review: multiplying and factoring polynomials

Lesson 02: Review: rational expressions, complex fractions

Lesson 03: Review: solving equations

Lesson 04: Review: equations of linear functions (lines)
Linear regression review

Lesson 05: Review: solutions of linear systems

Unit 1 review
Test: Unit 1 test

Unit 2: Basic trigonometry

Lesson 01: Angle conventions; definitions of the six trig functions

Lesson 02: Angle units; degrees(minutes & seconds), radians

Lesson 03: Given one trig ratio, find the others

Lesson 04: Special angles (0^o, 30^o, 60^o, 45^o, 90^o, 180^o, 270^o, 360^o)

Lesson 05: Evaluating trig function on the graphing calculator

Unit 2 review
Test: Unit 2 test

Unit 3: Triangle solutions

Lesson 01: Abstract solutions of right triangles

Lesson 02: Right triangle word problems, triangle area

Lesson 03: Vectors

Lesson 04: Sine law, more triangle area formulas

Lesson 05: Ambiguous case of the sine law

Lesson 06: Cosine law

Cumulative review, unit 3
Unit 3 review
Test: Unit 3 test

Unit 4: Trig identities

Lesson 01: Reciprocal and Pythagorean identities, trig simplifications

Lesson 02: Trig proofs

Lesson 03: Cosine composite angle identities

Lesson 04: Sine composite angle identities

Lesson 05: Tangent composite angle identities

Lesson 06: Product and factor identities, reference angles

Cumulative review, unit 4
Unit 4 review
Unit 4 test

Unit 5: Solving trig equations

Lesson 01: Simple trig equations

Lesson 02: Advanced trig equations

Unit 5 test

Unit 6: Function fundamentals

Lesson 1: Basic function definitions

Lesson 2: More on domain, intercepts, notation, function values

Lesson 3: Function operations, composite functions

Lesson 4: Reflections

Lesson 5: Even and odd functions

Lesson 6: Transformations of functions

Lesson 7: Minimum and maximum

Cumulative review, unit 6
Unit 6 review
Unit 6 test

Unit 7: Quadratic functions (parabolas)

Lesson 1: Transformations of quadratic functions

Lesson 2: Three forms of quadratic functions

Lesson 3: Quadratic calculator applications

Lesson 4: Quadratic area applications

Cumulative review, unit 7
Unit 7 review
Unit 7 test

Unit 8: Special functions

Lesson 1: Square root and semicircle functions

Lesson 2: Absolute value functions (|f(x)| and f(|x|) reflections)

Lesson 3: Piecewise functions, continuity

Lesson 4: Rate of change, piecewise word problems

Lesson 5: Greatest integer function, power functions

Cumulative review, unit 8
Unit 8 review
Unit 8 test

Unit 9: Polynomial functions

Lesson 1: Polynomial fundamentals, roots, end behavior

Lesson 2. Creating polynomial functions

Lesson 3: Long division (factoring, finding roots)

Lesson 4: Analyzing polynomials with a graphing calculator

Lesson 5: Applications (maximizing volume)

Cumulative review, unit 9
Unit 9 review
Unit 9 test

Semester summary
Semester review
Semester test

Enrichment Topics

Topic A: Analysis of absolute value inequalities

Topic B: Linear Programming

Topic C: Point-slope and intercept forms of a line

Topic D: The summation operator, Σ

Topic E: An unusual look at probability

Topic F: Rotations

Topic G: Absolute value parent functions

Topic H: Dimension changes affecting perimeter, area, and volume

Topic J: Algebraic solution to quadratic systems of equations.

Topic K: Derivation of the sine law

Topic L: Derivation of the cosine law

Topic M: Tangent composite function derivations

Topic N: Locating the vertex of a standard-form parabola

Topic O: Algebraic manipulation of inverse trig functions

Topic P: Logarithm theorem derivations

Topic Q: Arithmetic and geometric sum formulas

Topic R: Converting general form conics into standard form (completing-the-square)

Topic S: Conic section applications

Pre Calculus, Unit 1

Algebra Review

Unit 1: Lesson 01 Review: multiplying and factoring polynomials

Basic formulas that should be memorized:

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

$(a-b)(a+b) = a^2 - b^2$

$A^3 \pm B^3 = (A \pm B)(A^2 \mp AB + B^2)$

Example 1: Multiply $(3x - 2y)(3x + 2y)$

$(a - b)(a + b) = a^2 - b^2$

$(3x - 2y)(3x + 2y)$

$= \boxed{9x^2 - 4y^2}$

Example 2: Factor $x^2 - 9$

$a^2 - b^2 = (a-b)(a+b)$

$x^2 - 9 = \boxed{(x-3)(x+3)}$

Example 3: Square $(x - 6z)^2$

$(a-b)^2 = a^2 - 2ab + b^2$

$(x - 6z)^2 = \boxed{x^2 - 12xz + 36z^2}$

Example 4: Factor $x^2 + 8x + 16$

$a^2 + 2ab + b^2 = (a+b)^2$

$x^2 + 8x + 16 = \boxed{(x+4)^2}$

Example 5: Multiply $(x - 5y)(x^2 + 5xy + 25y^2)$

$(a-b)(a^2 + ab + b^2) = a^3 - b^3$

$(x - 5y)(x^2 + 5xy + 25y^2)$

$= x^3 - (5y)^3$

$= \boxed{x^3 - 125y^3}$

Example 6: Factor $27x^3 - z^3$

$(a)^3 - (b)^3$

$27x^3 - z^3 = (3x)^3 - (z)^3$

$= \boxed{(3x - z)(9x^2 + 3xz + z^2)}$

$(a - b)(a^2 + ab + b^2)$

Example 7: Multiply using FOIL (first, outside, inside, last). $(2x – 5) (x +3)$

$$(2x-5)(x+3) = \overset{F}{2x^2} + \overset{O}{6x} \overset{I}{-5x} \overset{L}{-15}$$

$$= \boxed{2x^2 + x - 15}$$

When factoring, always look for a greatest common factor (GCF).

Example 8: Factor $5x^2 + 5x – 30$

GCF = 5

$$5x^2 + 5x - 30 = 5(x^2 + x - 6)$$

$$= \boxed{5(x+3)(x-2)}$$

Assignment: In problems 1-8, perform the indicated multiplications.

1. $(y - x)(y + 11x)$

2. $3(x - 4)(x + 4)$

3. $(A - 3B)^2$

4. $(4mb + 9a)^2$

5. $(j + k)(j^2 - jk + k^2)$

6. $(5x + 20d)(5x - 20d)$

7. $(2m + 8n)(m + n)$

8. $(p^2 + 4h)^2$

Factor the given polynomial in the following problems:

9. $x^2 – 6x + 9$

10. $2x^2 – 4x – 48$

11. $3x^2 – 48$

12. $y^2 – 18y + 81$

13. $v^2 + 12v + 36$

14. $9f^2 - 100v^2$

15. $d^3 - c^3$

16. $54x^3 - 2y^3$

17. $x^2 + 12x + 35$

18. $x^2 - 10x - 24$

Unit 1: Lesson 02 Review: rational expressions, complex fractions

Rational expressions are of the form $\frac{P(x)}{Q(x)}$ where P(x) and Q(x) are polynomials. Rational expressions can be added, subtracted, multiplied, and divided.

Example 1: Find a common denominator and add or subtract as appropriate:

$$\frac{5}{x^2-9}-\frac{x-1}{x^2+x-12}$$

$$=\frac{5}{(x-3)(x+3)}-\frac{x-1}{(x+4)(x-3)}$$

$$=\frac{5}{(x-3)(x+3)}\,\frac{x+4}{x+4}-\frac{x-1}{(x+4)(x-3)}\,\frac{x+3}{x+3}$$

$$=\frac{5x+20-x^2-2x+3}{(x-3)(x+3)(x+4)}=\boxed{\frac{-x^2+3x+23}{(x-3)(x+3)(x+4)}}$$

Example 2: Simplify $\frac{x^2-16}{x+5}\;\frac{x^2+10x+25}{x^2+x-12}$

$$=\frac{(x-4)\cancel{(x+4)}}{\cancel{x+5}}\;\frac{(x+5)^{\cancel{2}}}{\cancel{(x+4)}(x-3)}=\boxed{\frac{(x-4)(x+5)}{x-3}}$$

Example 3: Simplify $\frac{x^3-1}{x-5}\div\frac{x^2-1}{x^2-8x+15}$

$$=\frac{\cancel{(x-1)}(x^2+1x+1)}{\cancel{(x-5)}}\;\frac{(x-3)\cancel{(x-5)}}{\cancel{(x-1)}(x+1)}$$

$$=\boxed{\frac{(x^2+x+1)(x-3)}{x+1}}$$

Example 4: Simplify this complex fraction. $\dfrac{\frac{2}{z^2}-\frac{5}{mz}-\frac{3}{m^2}}{\frac{2}{z^2}+\frac{7}{mz}+\frac{3}{m^2}}$

$$\frac{\frac{2}{z^2}-\frac{5}{mz}-\frac{3}{m^2}}{\frac{2}{z^2}+\frac{7}{mz}+\frac{3}{m^2}}\cdot\frac{m^2z^2}{m^2z^2}=\frac{\frac{2m^2z^2}{z^2}-\frac{5m^2z^2}{mz}-\frac{3m^2z^2}{m^2}}{\frac{2m^2z^2}{z^2}+\frac{7m^2z^2}{mz}+\frac{3m^2z^2}{m^2}}$$

$$=\frac{2m^2-5mz-3z^2}{2m^2+7mz+3z^2}$$

$$=\frac{\cancel{(2m+z)}(m-3z)}{\cancel{(2m+z)}(m+3z)}=\boxed{\frac{m-3z}{m+3z}}$$

Assignment: In problems 1-4, simplify by finding a common denominator and combining into one fraction.

1. $\frac{3}{x}+\frac{2}{y}$

2. $\frac{4}{8x^2}-\frac{3}{4xy}$

3. $\frac{3y}{y^2-5y+6}+\frac{4y}{y^2-4}$

4. $\frac{3}{x^2+3x+2}+\frac{x-3}{x^2-2x-3}$

5. Simplify $\frac{x^3-8}{x^2-4}\ \frac{x+3}{x^2-x-12}$

6. Simplify $\frac{2x^2-3x-2}{4x^2-1} \div \frac{x^2-4}{2x^2-5x+2}$

7. Simplify $\dfrac{\frac{2}{x-3}+\frac{-1}{x^2-9}}{\frac{5}{x^3-27}}$

8. Simplify $\dfrac{\frac{1}{x-3}}{\frac{x-5}{2x^2-4x-6}+3}$

Unit 1: Lesson 03 Review: solving equations

Solve these problems for the given variables:

Example 1: $\frac{2x}{3} - \frac{1}{5}(x-4) = 5$

$\frac{2x}{\cancel{3}}\,\cancel{15}^{5} - \frac{1}{\cancel{5}}\,\cancel{15}^{3}(x-4) = 5(15)$

$10x - 3(x-4) = 75$

$10x - 3x + 12 = 75$

$7x = 75 - 12 = 63$

$x = \frac{63}{7} = \boxed{9}$

Example 2: $3x^2 - 75 = 0$

$3(x^2 - 25) = 0$

$3(x-5)(x+5) = 0$

$x - 5 = 0 \qquad x + 5 = 0$

$x = \boxed{5} \qquad x = \boxed{-5}$

Example 3: $x^2 - x - 42 = 0$

$(x-7)(x+6) = 0$

$x - 7 = 0 \qquad x + 6 = 0$

$x = \boxed{7} \qquad x = \boxed{-6}$

Example 4: $\frac{1}{x-2} = \frac{x}{3}$

$\frac{1}{x-2} \times \frac{x}{3}$ cross multiply

$x(x-2) = 3$

$x^2 - 2x = 3$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x - 3 = 0 \qquad x + 1 = 0$

$x = \boxed{3} \qquad x = \boxed{-1}$

Example 5: $3x^3 - x^2 - 12x + 4 = 0$

$[x^2(3x-1) - 4(3x-1)] = 0$

$(3x-1)[x^2 - 4] = 0$

$(3x-1)(x-2)(x+2) = 0$

$3x - 1 = 0 \qquad x - 2 = 0 \qquad x + 2 = 0$

$3x = 1 \qquad x = \boxed{2} \qquad x = \boxed{-2}$

$x = \boxed{\frac{1}{3}}$

Example 6: $\frac{3}{x+2} - \frac{1}{x} = \frac{1}{5x}$

$$\frac{3}{\cancel{x+2}} \cdot \frac{5x\cancel{(x+2)}}{1} - \frac{1}{\cancel{x}} \cdot \frac{5\cancel{x}(x+2)}{1} = \frac{1}{\cancel{5x}} \cdot \frac{\cancel{5x}(x+2)}{1}$$

$$15x - 5(x+2) = x+2$$

$$15x - 5x - 10 = x + 2$$

$$10x - x = 2 + 10$$

$$9x = 12$$

$$x = \frac{12}{9} = \boxed{\frac{4}{3}}$$

Example 7: $\sqrt{x+4} = 7$

$$(\sqrt{x+4})^2 = (7)^2$$

$$x + 4 = 49$$

$$x = 49 - 4$$

$$x = \boxed{45}$$

check: (Because we raised to an even power)

$$\sqrt{45+4} = 7$$

$$\sqrt{49} = 7$$

$$7 \stackrel{\checkmark}{=} 7$$

Assignment: Solve for the variables.

1. $4 + \frac{x}{4} = \frac{2(x-2)}{5}$

2. $5x - 3 = 7x + 1$

3. $x^2 + 9x + 18 = 0$

4. $x^2 - 9 = 0$

5. $2x^2 + 10x + 3x + 15 = 0$

6. $5x^2 - 5x + x - 1 = 0$

7. $\frac{56}{x-2} = x - 3$

8. $(3/2)x + .5x + 7 = 0$

9. $x^2 = 81$

10. $2x^3 - 5x^2 - 12x = 0$

11. $\sqrt{x-2} = 6$

12. $\frac{3}{2x} + \frac{11}{5x} = 1$

Unit 1: Lesson 04

Review: equations of linear functions (lines)
Linear regression review

There are four forms of the equation of a line (a linear function):

- Ax + By + C = 0, **General** form (some textbooks put C on the right side of the equation)
- y = mx + b, **Slope-intercept** form
- $y - y_1 = m(x - x_1)$, **Point-slope** form
- $\frac{x}{a} + \frac{y}{b} = 1$, **Intercept** form

In the above equations *m* is the slope, (x_1, y_1) and (x_2, y_2) are points on the line, *a* is the x-intercept, and *b* is the y-intercept.

Notice in all forms both **x and y are to the one power** (degree 1).

Special cases:

- Equations of **vertical lines** are always of the form x = a where *a* is the place on the x-axis through which the line passes.
- Equations of **horizontal lines** are always of the form y = b where *b* is the place on the y-axis through which the line passes.

Example 1: Find the equation of the line (in slope-intercept form) passing through the points (4, 3) and (1, –6).

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-6-3}{1-4} = 3$

$y = mx + b$; $y = 3x + b$

$3 = 3(4) + b$ ← sub in (4, 3)

$3 - 12 = b$

$-9 = b$

$y = mx + b$

$\boxed{y = 3x - 9}$

Example 2: Find the equation of the line (in slope-intercept form) parallel to the line given by 3x + 2y = –1 and having y-intercept 5.

$3x + 2y = -1$

$2y = -3x - 1$

$y = -\frac{3}{2}x - \frac{1}{2}$; $m = -3/2$

y-intc = 5 → b = 5

$y = mx + b$

$\boxed{y = -\frac{3}{2}x + 5}$

Example 3: Express y = (3/4)x – 7 in general form.

$y = \frac{3}{4}x - 7$

$4y = \frac{3}{4}x(4) - 7\cdot 4$

$4y = 3x - 28$

$\boxed{-3x + 4y + 28 = 0}$

Example 4: Express y = (3/4)x – 7 in intercept form.

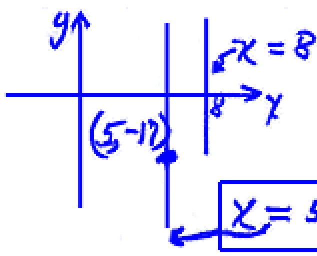
$-\frac{3}{4}x + y = -7 \leftarrow$ need 1 here

$-\frac{3x}{4(-7)} + \frac{y}{-7} = \frac{-7}{-7}$ Divide by -7

$\frac{3x}{28} + \frac{y}{-7} = 1$

$\boxed{\frac{x}{28/3} + \frac{y}{-7} = 1}$

Example 5: Find the equation of the line parallel to the line x = 8 and passing through the point (5, –17).



$\boxed{x = 5}$

Example 6: Find the equation of the line (in point-slope form) passing through (–1, 6) and perpendicular to the line given by 3x + 7y = –9.

$3x + 7y = -9$

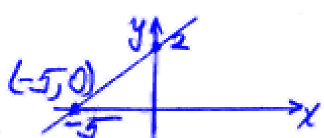
$7y = -3x - 9$

$m = -\frac{3}{7}$; $y = -\frac{3}{7}x - 9/7$

$m_\perp = \frac{7}{3}$; $y - y_1 = m(x - x_1)$

$\boxed{y - 6 = \frac{7}{3}(x + 1)}$

Example 7: Find the equation of the line (in intercept form) that passes through (–5, 0) and has y-intercept 2.



$a = -5$; $b = 2$

$\frac{x}{a} + \frac{y}{b} = 1$

$\boxed{\frac{x}{-5} + \frac{y}{2} = 1}$

Example 8: Find the equation of the line (in slope-intercept form) that has x-intercept –8 and y-intercept 9.

y-intc → $b = 9$

x-intc → $(-8, 0)$

$y = mx + b$; $y = mx + 9$

$0 = m(-8) + 9$; sub in $(-8, 0)$

$8m = 9$; $m = 9/8$

$y = mx + b$

$\boxed{y = \frac{9}{8}x + 9}$

Recall from Alg II, that scattered points (a scatter plot) can often be approximately fit with a straight line. The process of finding the line of best fit is known as **linear regression**.

For a review on this process, see **Calculator Appendix M** (scatter plots) and **Calculator Appendix N** (regression). Note that with regression it is also possible for other than linear functions (exponential, logarithm, etc.) to be made to fit scatter plots.

Take special note of the **correlation factor, *r*,** given by a regression which is basically a score of how good the fit is. The value of *r* is restricted to $-1 \le r \le 1$. The closer $|r|$ is to 1, the better the fit.

When the best fit line has a negative slope, the correlation is said to be negative. Likewise, when the slope is positive, the correlation is said to be positive.

Assignment:

1. Find the equation of the line (in intercept form) that has x-intercept 2 and y-intercept –7.

2. Find the equation of the line (in slope-intercept form) that passes through (5,6) and (1, –4).

3. Express $3x + 7y = 9$ in intercept form.

4. Express $3x + 7y = 9$ in slope-intercept form.

5. Write the equation of the line that is perpendicular to the x-axis and passing through the point (5, 1).

6. Write the equation of the line (in slope-intercept form) passing through (4, 6) and intersecting the y-axis at 17.

7. Write the equation of the line (in general form) that is perpendicular to the line given by $x = 2y - 6$ and passing through (101, 79).

8. Write the equation of the line (in point-slope form) having slope 5 and passing through (– 6, 2).

9. Write the equation of the line (in slope-intercept form) having slope 5 and passing through (– 6, 2).

10. Write the equation of a line having slope 2 and y-intercept 19.

11. What is the slope of a line perpendicular to $x/4 - y/7 = 1$?

12. What is the equation of the line parallel to the line given by y + 2 = 0 and passing through (1.5, 3/4)?

13. What is the equation of the line parallel to the line given by x + 2 = 0 and passing through (1.5, 3/4)?

14. Find the equation of the line passing through (2, 5) and (– 2, 1) in point slope form.

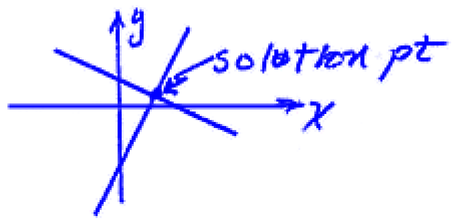
15. Perform a linear regression on this data and show the equation of the best-fit line along with a sketch of the line and scatter-plot. Give the correlation coefficient.

x	**y**
–4.0	9.0
–1.1	4.2
2.0	–0.5
1.9	–2.0
6.0	–5.0
5.8	–7.0

Unit 1: Lesson 05 **Review: solving linear systems**

Solving the following "system" of linear equations is equivalent to finding the (x, y) **intersection point** of the two lines:

$y = 3x - 6$
$y = -x + 4$



The following example demonstrates the solution of a linear system of equations using the method of **elimination**.

Example 1: $x + y = -1$
$3x - 2y = 11$

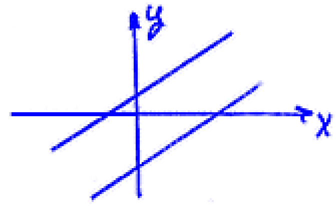
$2(x+y) = -1(2) \longrightarrow 2x + \cancel{2y} = -2$
$3x - 2y = 11 \longrightarrow 3x - \cancel{2y} = 11$
$5x = 9$
$x = \boxed{\frac{9}{5}}$
$x + y = -1$
$\frac{9}{5} + y = -1$
$y = -1 - \frac{9}{5} = \frac{-5}{5} - \frac{9}{5} = \boxed{\frac{-14}{5}}$

The following example demonstrates the solution of a linear system of equations using the method of **substitution**.

Example 2: $x + y = -1$
$3x - 2y = 11$

$\longrightarrow y = -x - 1$
$3x - 2(-x - 1) = 11$
$3x + 2x + 2 = 11$
$5x = 9$
$x = \boxed{\frac{9}{5}}$
$x + y = -1$
$\frac{9}{5} + y = -1$
$y = -\frac{9}{5} - 1$
$y = \boxed{\frac{-14}{5}}$

When algebraically solving a linear system, equations are sometimes produced that are obviously not true. This means the equations of the system are **inconsistent** and **independent** and there is no solution. This happens when the two **lines are parallel and separated**, and as a result there is no intersection point.



No solution

Notice in the following example that a nonsense equation is produced, thus indicating no solution.

Example 3: $4x - y = -2$
$12x - 3y = 5$

$$-3(4x - y) = -2(-3) \rightarrow -\cancel{12x} + \cancel{3y} = +6$$
$$12x - 3y = 5 \rightarrow \cancel{12x} - \cancel{3y} = 5$$
$$0 \neq 11$$
$$\boxed{\text{No solution}}$$

Occasionally during the solution of a linear system a true equation is produced that does not lead to a solution. In this case the equations are **consistent** and **dependent**. The two lines are **right on top of each other** producing an infinite number of intersection points along the lines.

This is illustrated in the following example.

Example 4: $y = 2x - 5$ and $-6x + 3y = -15$

$$y = 2x - 5 \qquad -6x + 3(2x - 5) = -15$$
$$-\cancel{6x} + \cancel{6x} - 15 = -15$$
$$-15 = -15$$
$$\boxed{\begin{array}{l}\text{Infinite \# of solution pts} \\ \text{along the line } y = 2x - 5\end{array}}$$

See **Enrichment Topic I** for how to solve three equations for three variables.

See **Enrichment Topic J** for how to solve quadratic systems of equations.

Assignment:

In the following problems, solve for the intersection point(s) of the systems of linear equations using the **substitution** method:

1. $y = 8x - 11 \quad and \quad x - 2y = 1$

2. $\mathrm{x} = -(1/3)\mathrm{y} + 4$ and $\mathrm{y} = -3\mathrm{x} + 1$

3. $\mathrm{x} + \mathrm{y} = 6$ and $2\mathrm{y} + 2\mathrm{x} = 0$

4. $\left(\frac{1}{2}\right)\mathrm{y} = \left(\frac{1}{3}\right)\mathrm{x} + 2$ and x – y – 1 = 0

5. $\mathrm{x} = 1$ and $\mathrm{y} = \mathrm{x} + 2$

6. $18x - .5y = 7 \quad and \quad y = 8$

In the following problems, solve for the intersection point(s) of the systems of linear equations using the **elimination** method:

7. $\mathrm{x} + \mathrm{y} = 5 \quad \text{and} \quad -\mathrm{x} + 11\mathrm{y} = 0$

8. $4 = \mathrm{x} + 2\mathrm{y} \quad \text{and} \quad -2\mathrm{y} + \mathrm{x} = 1$

9. $-3\mathrm{x} + \mathrm{y} = 1 \;\text{ and }\; 18\mathrm{x} - 6\mathrm{y} = 1$

10. $\mathrm{x} + \mathrm{y} + 1 = 0 \quad \text{and} \quad 2\mathrm{x} - 9\mathrm{y} = 3$

11. $x = 2 \quad and \quad x - 5y = 2$

12. $\mathrm{y} = 5\mathrm{x} - 6 \;\text{ and} - 15\mathrm{x} + 3\mathrm{y} = -18$

In the following problems, use any technique. Hint: Make a sketch of the two lines in which case you might be able to "see" the answer. This is called solving by "inspection."

13. $x = -5 \quad and \quad y = 11$	**14.** $x + 2 = 0 \quad and \quad 2(8 - y) = 0$

Unit 01: Review

1. Multiply $(3x - 9)(5x + 2)$

2. Factor $5x^2 - 45$

3. Factor $x^2 + 4x - 21$

4. Factor $8x^3 - 27$

5. Simplify $\frac{x^2-16}{x-3}\frac{2x-6}{x^2+10x+24}$

6. Combine into a single fraction and simplify: $\frac{x^3-1}{x^2+x+1}+\frac{16}{2x^2+18x-20}$

7. Simplify $\dfrac{\frac{5}{x-5}}{\frac{7}{2x}+\frac{x+1}{x^2-25}}$

8. Expand $(2x-8)^2$

9. Expand $2(x+9y)^2$

10. Solve $\frac{2}{x-6} + 3 = 11$

11. Solve $2x^2 + 12x + 10 = 0$

12. Solve $3x^2 - 7x + 6x - 14 = 0$

13. Solve $t^3 - 36t = 0$

14. Solve $\sqrt{x+3} = -4$

15. Find the equation (in slope-intercept form) of the line passing through (5, 1) and (–6, 2).

16. Find the equation (in point-slope form) of the line perpendicular to the line given by $x + 5y = 19$ and passing through (11, 8).

17. Find the equation (in intercept form) of the line passing through (0, 8) and having slope –3.

18. What is the equation of the line perpendicular to the x-axis and passing through (11, 3)?

19. What is the equation of the horizontal line passing through (9, –16)?

20. Find the intersection point of these two lines using substitution:
$x + y = 18$ and $4x - y = 12$

21. Find the intersection point of these two lines using elimination:
$x + y = 18$ and $4x - y = 11$

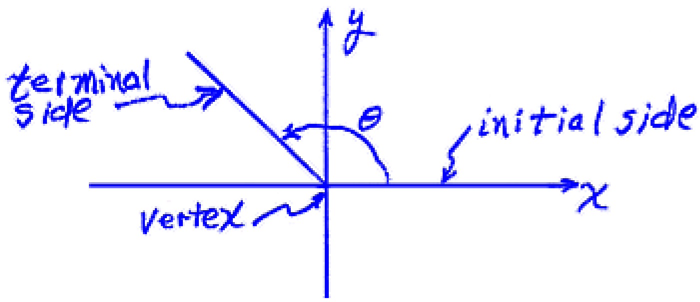
Pre Calculus, Unit 2

Basic Trigonometry

Unit 2: Lesson 01 Angle conventions, definitions of the six trig functions

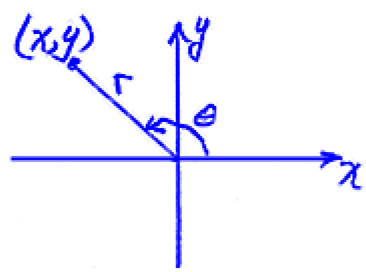
The standard position of an angle throughout all mathematics (not just trig) has its **vertex at the origin** and **initial side on the positive x-axis**. The other side is called the **terminal side**.

Note that **positive angles** rotate **counter clock-wise**.



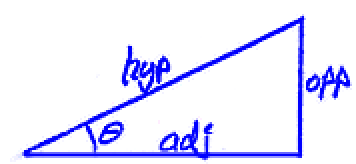
Three definitions of the sine (abbreviated sin), cosine (abbreviated cos), and tangent (abbreviated tan) of an angle (we will call our angle by the Greek letter theta, θ):

First definition (x, y, & r):
Begin with an angle in standard position and with the terminal side in the 2nd quadrant (it could be in any quadrant).



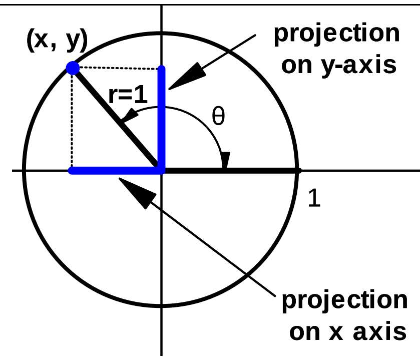
$\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$ $\tan\theta = \frac{y}{x}$

Second definition (opp, adj, hyp):
(opposite, adjacent, hypotenuse)
Let θ be one of the acute angles of a right triangle.



$\sin\theta = \frac{opp}{hyp}$ $\cos\theta = \frac{adj}{hyp}$ $\tan\theta = \frac{opp}{adj}$

Third definition (projection):
Draw an angle in standard position with a circle centered at the origin with radius 1 (called a unit circle).



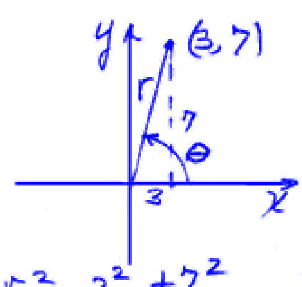
$\sin\theta =$ projection on y axis
$\cos\theta =$ projection on x-axis
$\tan\theta \rightarrow$ No proj definition

Defining the other three trig functions:

The basic fact to remember here is that cosecant (csc), secant (sec), and cotangent (cot) are **reciprocals** respectively of sin, cos, and tan.

$\sec\theta = \frac{r}{x} = \frac{hyp}{adj} \longrightarrow$ reciprocal of cos
$\csc\theta = \frac{r}{y} = \frac{hyp}{opp} \longrightarrow$ " " sin
$\cot\theta = \frac{x}{y} = \frac{adj}{opp} \longrightarrow$ " " tan

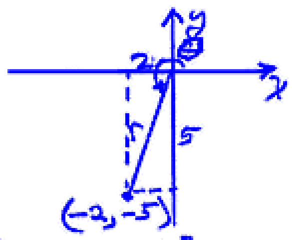
Example 1: Draw the angle, θ, in standard position where (3, 7) lies on the terminal side. Find the values of all six trig functions.



$r^2 = 3^2 + 7^2$
$r = \sqrt{9+49} = \sqrt{58}$

$\sin\theta = \frac{y}{r} = \frac{7}{\sqrt{58}}$
$\cos\theta = \frac{x}{r} = \frac{3}{\sqrt{58}}$
$\tan\theta = \frac{y}{x} = \frac{7}{3}$
$\cot\theta = \frac{x}{y} = \frac{3}{7}$
$\sec\theta = \frac{r}{x} = \frac{\sqrt{58}}{3}$
$\csc\theta = \frac{r}{y} = \frac{\sqrt{58}}{7}$

Example 2: Draw the angle, θ, in standard position where (–2, –5) lies on the terminal side. Find the values of all six trig functions.



$r^2 = 2^2 + 5^2$

$r = \sqrt{4+25} = \sqrt{29}$

$\sin\theta = \frac{y}{r} = \frac{-5}{\sqrt{29}}$

$\cos\theta = \frac{x}{r} = \frac{-2}{\sqrt{29}}$

$\tan\theta = \frac{y}{x} = \frac{-5}{-2} = \frac{5}{2}$

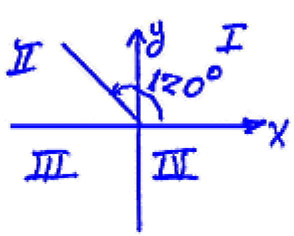
$\cot\theta = \frac{x}{y} = \frac{-2}{-5} = \frac{2}{5}$

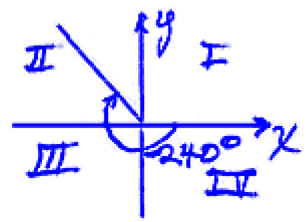
$\sec\theta = \frac{r}{x} = \frac{\sqrt{29}}{-2}$

$\csc\theta = \frac{r}{y} = \frac{\sqrt{29}}{-5}$

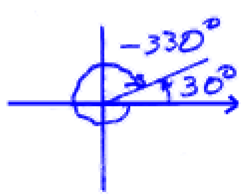
Co-terminal angles have a common terminal side. Naturally, the values of all six trig functions of an angle are the same as those angles with which it is co-terminal.

Example 3: Draw a 120° angle in standard position. Then draw a negative angle in standard position that is co-terminal with the 120° angle. Identify the four quadrants.





Example 4: If sin(30°) = .5 what is csc(–330°)?



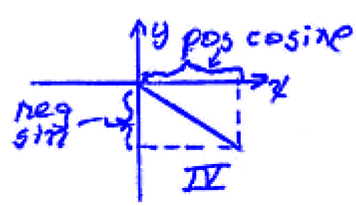
These angles are co-terminal so they have the same sin.

$\sin(-330) = .5$

$\csc(-330) = \frac{1}{.5} = \boxed{2}$

reciprocal

Example 5: If the sine of an angle is negative and the cosine is positive, in what quadrant is the angle?



$\boxed{\text{4th quadrant}}$

Assignment: In problems 1-4, draw the angle in standard position and identify the quadrant of the terminal side.

1. 50^o

2. 350^o

3. 130^o

4. -120^o

5. Define all six trig functions in terms of x, y, and r.

6. Define all six trig functions in terms of opp, adj, and hyp.

7. A straight stick protrudes out of the ground exactly 1 meter. As a result of the sun being directly overhead, the shadow on the ground of the slightly leaning stick is exactly .32 meters. What is the cosine of the angle between the stick and the horizontal ground?

8. Draw an angle in standard position in which the projection of the terminal side is positive on the y-axis and negative on the x-axis. In which quadrant is the terminal side?

9. Draw an angle, θ, in standard position whose terminal side includes the point (5, 6). Find sin θ, cos θ, and tan θ.

10. Draw an angle, θ, in standard position whose terminal side includes the point (1, –2). Find sin θ, cos θ, and tan θ.

11. Draw an angle, θ, in standard position whose terminal side includes the point (–4, –3). Find csc θ, sec θ, and cot θ.

12. Draw two angles in standard position (but each on its own coordinate system) that are co-terminal. The terminal side of both angles should be in the third quadrant.

13. Inside a circle of radius 1, draw a 120^{o} angle in standard position. Show the projection of this angle on the two axes. What conclusion can be reached regarding the sign of the sine and cosine of 120^{o}?

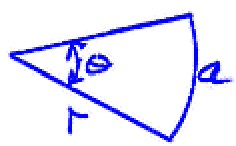
14. If tan θ = 11.2 what would be the value of cot θ?	15. If the sine and cosine of an angle are both negative, in what quadrant is the angle? Draw the angle.

Unit 2: Lesson 02 — Angle units; degrees(minutes & seconds), radians

Here are two units of measure for angles with which we should be familiar.

- Degrees … 360^{o} in a full circle.
- Radians … 2π(radians) ≈ 6.28 in a full circle

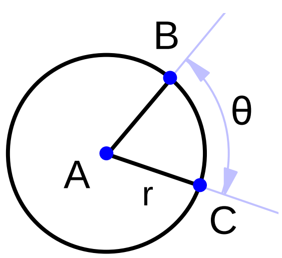
The definition of radians comes from a drawing and an associated formula:



$$\theta = \frac{a}{r}$$

In the drawing above, *r* is the radius, θ is the **angle in radians**, and, *a* is the arc length.

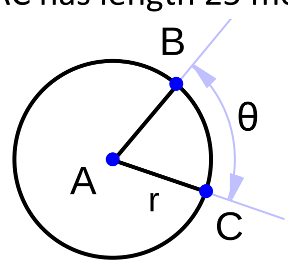
Example 1: In the drawing below the arc length BC is 20 ft and the radius is 18 ft. Find the measure of θ in radians.



$$\theta = \frac{BC}{r}$$

$$\theta = \frac{20}{18} = \boxed{1.\overline{1} \text{ radians}}$$

Example 2: In the drawing below find the length of the arc BC when the angle θ is π/3 radians and the line segment AC has length 25 meters.



$$\theta = \frac{BC}{r}$$

$$\frac{\pi}{3} = \frac{BC}{25}$$

$$3BC = 25\pi$$

$$BC = \frac{25\pi}{3} = \boxed{26.1799\,m}$$

Conversions between degrees and radians (**these must be memorized**):

$90^o = \pi/2$ radians $180^o = \pi$ radians

$270^o = 3\,\pi/2$ radians $360^o = 2\pi$ radians

For non-special angles we will use the conversion formula: $\frac{deg}{rad} = \frac{deg}{rad}$

For the left side of the equation we will use the correspondence of 180^o to π radians, so the formula becomes: $\frac{180}{\pi} = \frac{deg}{rad}$

Example 3: Convert 38^o into radians.

$$\frac{180}{\pi} = \frac{38}{\theta}$$

$$180\theta = 38\pi$$

$$\theta = \frac{38\pi}{180} = \boxed{.663225 \text{ rad.}}$$

Example 4: Convert $2\pi/7$ radians to degrees.

$$\frac{180}{\pi} = \frac{\theta}{2\pi/7}$$

$$\cancel{\pi}\theta = 180(2\cancel{\pi}/7)$$

$$\theta = \frac{360}{7} = \boxed{51.42857^o}$$

Each degree can be broken up into **minutes** and **seconds**. These are **angular measures**, **not time**, although the analogy is certainly there.

1 degree = 60 minutes

1 minute = 60 seconds

Example 5: Express an angle of 14 degrees, 57 minutes and 32 seconds in "shorthand" notation.

$14^\circ 57' 32''$

Example 6: Convert 123^o 5' 49'' into decimal form.

$$49'' = \frac{49}{60}' = .81\bar{6}'$$

$$5' + .81\bar{6}' = 5.81\bar{6}'$$

$$5.81\bar{6}' = \frac{5.81\bar{6}}{60}^\circ = .0969\bar{4}^\circ$$

$$123^\circ + .0969\bar{4}^\circ = \boxed{123.0969\bar{4}^\circ}$$

Example 7: Convert 238.458° into degrees, minutes, and seconds.

$$.458^\circ = .458(60)' = 27.48' = 27' + .48'$$

$$.48' = .48(60)'' = 28.8''$$

$$238.458^\circ = \boxed{238^\circ\ 27'\ 28.8''}$$

See **Calculator Appendix S** (and an associated video) for how to convert between degrees, minutes, seconds, and radians on a TI graphing calculator.

Assignment: In all conversions to degrees, do so in decimal form unless otherwise specifically instructed to use degrees, minutes, and seconds.

1. Convert 122^o to radians.

2. Convert $7\pi/12$ radians to degrees.

3. Convert $-\pi/6$ radians to degrees.

4. Convert 72^o to radians.

5. Convert 41 seconds into minutes.

6. Convert 19 minutes into degrees.

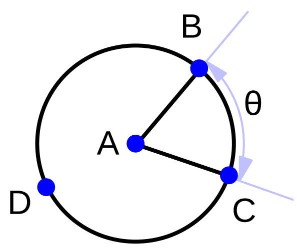
7. Convert $\pi/2$ into degrees.

8. Convert 270^o into radians.

9. Approximately how many degrees is one radian?

10. Draw a central angle of .5 radians with a radius of 5 out to the arc BC. What is the length of arc BC?

11. In the drawing below find the length of arc BDC when θ = 60^{o} and AC = 22.



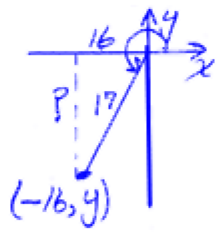
12. Convert 187.926^{o} into degrees, minutes, and seconds.

13. Convert 15^{o} 11' 46'' into decimal degrees.

Unit 2: Lesson 03 Given one trig ratio, find the others

In the following examples, point P (x, y) is on the terminal side of an angle in standard position. The distance from the vertex to P is *r*. Find the one of *x*, *y*, and *r* that is missing.

Example 1: x = –16, r = 17, P(–16, y) is in the third quadrant.



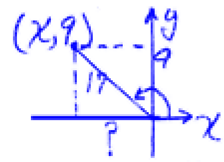
$r^2 = x^2 + y^2$

$y^2 = r^2 - x^2$

$y = \pm\sqrt{17^2 - (-16)^2} = \boxed{-5.74456}$

choose – because of 3rd quad.

Example 2: y = 9, r = 17, P(x, 9) is in the second quadrant.



$x^2 + y^2 = r^2$

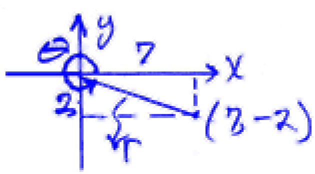
$x^2 = r^2 - y^2$

$x = \pm\sqrt{17^2 - 9^2}$

$x = \boxed{-14.4222}$

Choose – sign because of 2nd quad

Example 3: x = 7, y = –2



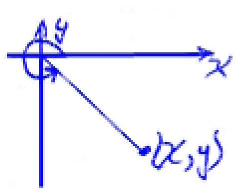
$r^2 = x^2 + y^2$

$r = \sqrt{7^2 + (-2)^2}$

$r = \sqrt{53} = \boxed{7.280109}$

In the following examples, draw an angle in standard position that satisfies the given conditions and then find the other five trig functions.

Example 4: cos θ = 5/13, θ in the fourth quadrant



$\cos\theta = \frac{5}{13} = \frac{x}{r}$

$x = 5 \quad r = 13$

$y = \pm\sqrt{r^2 - x^2}$

$y = \pm\sqrt{13^2 - 5^2}$

$y = -12$

$\sin\theta = \frac{y}{r} = \frac{-12}{13}$

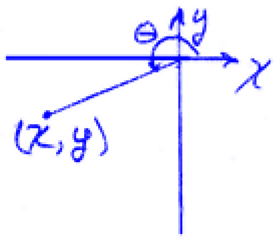
$\tan\theta = \frac{y}{x} = \frac{-12}{5}$

$\cot\theta = \frac{x}{y} = \frac{-5}{12}$

$\sec\theta = \frac{r}{x} = \frac{13}{5}$

$\csc\theta = \frac{r}{y} = \frac{-13}{12}$

Example 5: tan θ = 8/15, θ in the third quadrant



$\tan\theta = \frac{8}{15} = \frac{-8}{-15} = \frac{y}{x}$

$x = -15 \quad y = -8$

$r^2 = x^2 + y^2$

$r = \sqrt{(-15)^2 + (-8)^2}$

$r = 17$

$\sin\theta = \frac{y}{r} = \frac{-8}{17}$

$\cos\theta = \frac{x}{r} = \frac{-15}{17}$

$\cot\theta = \frac{x}{y} = \frac{-15}{-8} = \frac{15}{8}$

$\sec\theta = \frac{r}{x} = \frac{17}{-15}$

$\csc\theta = \frac{r}{y} = \frac{17}{-8}$

Assignment: In problems 1-4, point P (x, y) is on the terminal side of an angle in standard position. The distance from the vertex to P is *r*. Find the one of *x*, *y*, and *r* that is missing.

1. $x = 4$, $r = 6$, P(4, y) in the first quadrant

2. $y = -24$, $r = 25$, P(x, –24) in the fourth quadrant

3. $y = -5$, $r = 8$, P(x, –5) in the third quadrant

4. $x = -12$, $r = 20$, P(–12, y) in the second quadrant

In the following problems, draw an angle in standard position that satisfies the given conditions and then find the remaining unknown trig functions. When a point P is given it is assumed to be on the terminal side of the angle.

5. $\tan \theta = -3/4$, θ in the fourth quadrant

6. $\sec \theta = -17/8$, θ in the third quadrant

7. $\cot \theta = 5/3$, θ in the first quadrant

8. $\sin \theta = 5/6$, θ in the 2nd quadrant

9. $P(-3, y)$, $y > 0$, radius of 6

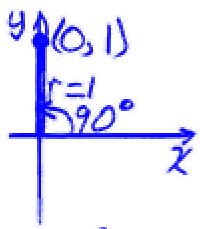
10. $P(x, -7)$, $x > 0$, radius of 12

Unit 2: Lesson 04 Special angles (0°, 30°, 60°, 45°, 90°, 180°, 270°, 360°)

The quadrantal angles are those that are integral multiples of 90° (0°, 180°, 270°, 360°). The easiest way to evaluate the trig function values of these angles is by using the *x*, *y*, *r* definitions.

In the following two examples, draw the terminal side of the angle in standard position containing point P with a radius of 1. Label the (x, y) values of this point. Then determine the values of sin, cos, and tan.

Example 1: 90°



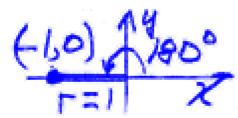
$x = 0$
$y = 1$
$r = 1$

$\sin 90 = \frac{y}{r} = \frac{1}{1} = \boxed{1}$

$\cos 90 = \frac{x}{r} = \frac{0}{1} = \boxed{0}$

$\tan 90 = \frac{y}{x} = \frac{1}{0}$ $\boxed{\text{undefined}}$

Example 2: 180°



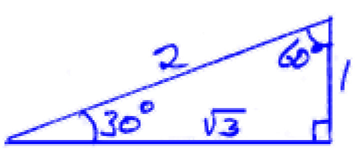
$x = -1$
$y = 0$
$r = 1$

$\sin 180 = \frac{y}{r} = \frac{0}{1} = \boxed{0}$

$\cos 180 = \frac{x}{r} = \frac{-1}{1} = \boxed{-1}$

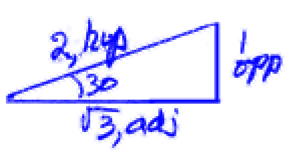
$\tan 180 = \frac{y}{x} = \frac{0}{-1} = \boxed{0}$

The angles 30° and 60° are special angles, both coming from a 30-60-90 triangle. Recall from geometry the following relationship between the sides of this special triangle (**must be memorized**):



prove with
$2^2 = 1^2 + (\sqrt{3})^2$

Example 3: Having previously memorized the six trig functions in terms of opp, adj, & hyp, these functions are easily determined for 30°.



$\sin 30 = \frac{opp}{hyp} = \boxed{\frac{1}{2}}$

$\cos 30 = \frac{adj}{hyp} = \boxed{\frac{\sqrt{3}}{2}}$

$\tan 30 = \frac{opp}{adj} = \boxed{\frac{1}{\sqrt{3}}}$

$\csc 30 = \frac{hyp}{opp} = \boxed{\frac{2}{1} = 2}$

$\sec 30 = \frac{hyp}{adj} = \boxed{\frac{2}{\sqrt{3}}}$

$\cot 30 = \frac{adj}{opp} = \boxed{\frac{\sqrt{3}}{1}}$

Example 4: Similarly the six trig functions of 60° are determined.

$\sin 60 = \frac{opp}{hyp} = \boxed{\frac{\sqrt{3}}{2}}$

$\cos 60 = \frac{adj}{hyp} = \boxed{\frac{1}{2}}$

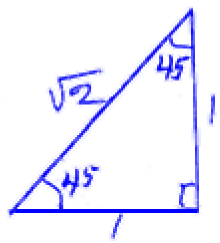
$\tan 60 = \frac{opp}{adj} = \frac{\sqrt{3}}{1} = \boxed{\sqrt{3}}$

$\cot 60 = \frac{adj}{opp} = \frac{1}{\sqrt{3}} = \boxed{\frac{\sqrt{3}}{3}}$

$\sec 60 = \frac{hyp}{adj} = \frac{2}{1} = \boxed{2}$

$\csc 60 = \frac{hyp}{opp} = \frac{2}{\sqrt{3}} = \boxed{\frac{2\sqrt{3}}{3}}$

The final special triangle we will consider is the 45-45-90 triangle. Recall from geometry the following relationship between the sides of this special triangle (**must be memorized**):



prove with
$(\sqrt{2})^2 = 1^2 + 1^2$

Example 5: Having previously memorized the six trig functions in terms of opp, adj, & hyp, these functions are easily determined for 45°.

opp & adj are both 1

hyp = $\sqrt{2}$

$\sin 45 = \frac{opp}{hyp} = \frac{1}{\sqrt{2}} = \boxed{\frac{\sqrt{2}}{2}}$

$\cos 45 = \frac{adj}{hyp} = \frac{1}{\sqrt{2}} = \boxed{\frac{\sqrt{2}}{2}}$

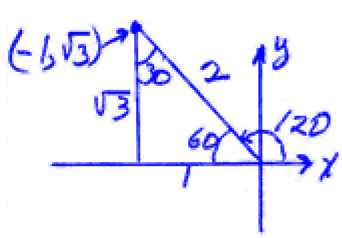
$\tan 45 = \frac{opp}{adj} = \frac{1}{1} = \boxed{1}$

$\cot 45 = \frac{adj}{opp} = \frac{1}{1} = \boxed{1}$

$\sec 45 = \frac{hyp}{adj} = \frac{\sqrt{2}}{1} = \boxed{\sqrt{2}}$

$\csc 45 = \frac{hyp}{opp} = \frac{\sqrt{2}}{1} = \boxed{\sqrt{2}}$

Example 6: Evaluate the sine, cosine, and tangent of 120°.



$\sin 120 = \frac{y}{r} = \boxed{\frac{\sqrt{3}}{2}}$

$\cos 120 = \frac{x}{r} = \boxed{\frac{-1}{2}}$

$\tan 120 = \frac{y}{x} = \boxed{\frac{\sqrt{3}}{-1} = -\sqrt{3}}$

Assignment: In problems 1-4, draw the terminal side of the angle in standard position containing point P with a radius of 1. Label the (x, y) values of this point. Then determine the values of sine, cosine, and tangent.

1. 90^{o}	2. 270^{o}
3. 360^{o}	4. 180^{o}
5. Draw a 30-60-90 triangle and label the standard lengths of the sides.	6. Draw a 45-45-90 triangle and label the standard lengths of the sides.

In problems 7-10, draw the angle in standard position. Draw in an appropriate 30-60-90 or 45-45-90 triangle. Using the triangle as a reference, label an (x, y) point on the terminal side of the angle. Using the *x*, *y*, *r* definitions, determine the sine, cosine, and tangent of the angle.

7. 300^o	8. 210^o
9. 135^o	10. 225^o

In the following problems, draw the angle in standard position. Draw in an appropriate 30-60-90 or 45-45-90 triangle. Using the triangle as a reference, label an (x, y) point on the terminal side of the angle. Using the *x*, *y*, *r* definitions, determine the secant, cosecant, and cotangent of the angle.

11. 150^o

12. 315^{o}

Unit 2: Lesson 05

Evaluating trig functions on the graphing calculator

See **Calculator Appendix T** (and an associated video) for how to evaluate any of the six trig functions on a graphing calculator.

Note that the calculator will only directly calculate sine, cosine, and tangent. Recall that the other three trig functions are simply reciprocals of these.

For example, to find cot 28°, enter 1/tan(28) into the calculator. (Make sure the **mode** is set to degrees for this problem.)

In the following examples, use a calculator to find the trig expression. Assume all angles are in radians unless the degree symbol (°) is used. Be sure to switch to the appropriate mode before entering each problem.

Example 1: cos(2π/7)

mode → rad

$\cos(2\pi/7) = \boxed{.6234898}$

Example 2: 1 – sec(238°)

mode → deg

$1 - 1/\cos(238)$

$= \boxed{2.8870799}$

Example 3: tan(–127°)

mode → deg

$\tan(-127)$

$= \boxed{1.3270448}$

Example 4: (14 + csc(π/5))/sin (π/7)

Mode → rad

$(14 + 1/\sin(\pi/5))/\sin(\pi/7)$

$= \boxed{36.18780839}$

Calculator Appendix S (and associated video) shows how to convert between decimal degree and degrees, minutes, & seconds.

In the following examples, use the graphing calculator to produce the desired conversion.

Example 5: Convert 128.784° to deg-min-sec form.

mode → deg
128.784 ▸DMS
$= \boxed{128^\circ 47' 2.4''}$

Example 6: Convert 45° 8' 22'' to decimal degrees form.

mode → deg
45° 8' 22'' Enter
$= \boxed{45.139\overline{4}}$

Assignment: In problems 1-6, use a calculator to find the trig expression. Assume all angles are in radians unless the degree symbol (^{o}) is used. Be sure to switch to the appropriate mode before entering each problem.

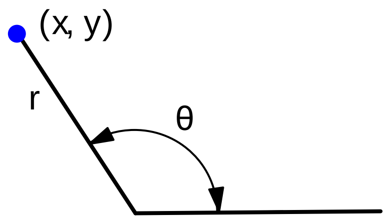
1. $\sin(48^o)$
2. $\cos(11\pi/3)$
3. $5 + \cot(2\pi/10)$
4. $\tan(-11.23^o) + 22$
5. $\sqrt{\cos(11°) + \sec(11°)}$
6. $56 - \csc(\pi - \pi/6)$

In the following problems, use the graphing calculator to produce the desired conversion.

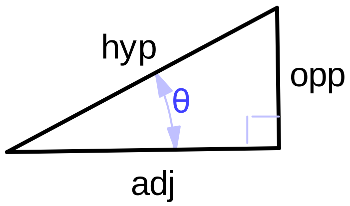
7. Convert 137^o 18' 29'' to decimal degrees form.	8. Convert 128.784^o to deg-min-sec form.

Unit 2: Review

1. Define the six trig functions in terms of *x*, *y*, and *r*.



2. Define the six trig functions of θ in terms of *opp*, *adj*, & *hyp*.



3. Define the angle θ in radians with a drawing and accompanying equation.

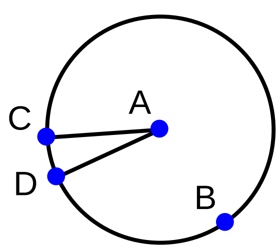
4. The sine of an angle drawn inside a unit circle (radius 1) is the projection of the radius on which axis?	5. The cosine of an angle drawn inside a unit circle (radius 1) is the projection of the radius on which axis?

6. Convert 272^{o} into radians.

7. Convert $7\pi/17$ into degrees.

8. How many radians are there in a full circle?	9. Since cot is not a function provided on a graphing calculator, how can it be done on the calculator?

10. Angle DAC is 32° and the radius of the circle is 4 ft. How long is arc DC?



11. Using the drawing in problem 10, and assuming the length of arc DBC is 28.3 ft. and the radius of the circle is 5 ft., what is the measure (in radians) of the angle that arc subtends?

In problems 12-13, point P (x, y) is on the terminal side of an angle in standard position. The distance from the vertex to P is *r*. Draw and label the angle and then find the one of *x*, *y*, and *r* that is missing.

12. x = –12, r= 17, P(–12, y), is in the 2nd quadrant.	13. y = –10, r= 15.92, P(x, –10), is in the 3rd quadrant.

14. List the sine and cosine of the quadrantal angles (90^o, 180^o, 270^o, 360^o).

15. Draw a 30-60-90 triangle and label the lengths of the sides. Using these sides give sin 30^o, cos 60^o, sec 30^o, cot 60^o.

16. Given point P (5, –7) on the terminal side of an angle θ in standard position, determine the cotangent and cosecant of θ.

---

17. Given point P (–3, 8) on the terminal side of an angle θ in standard position, determine the sine and cosine of θ.

---

18. Determine the sine and cosine of $11\pi/6$ radians.

---

19. Determine the secant and tangent of 210^{o}.

20. Draw a 45-45-90 triangle and label the lengths of the sides. Using these sides give the sine, cosine, and tangent of $3\pi/4$ radians.

21. Use a calculator to evaluate the following expression. The angles are in radians.

$$147.2(\sin(5\pi/3) - \csc(14.3\pi))$$

Pre Calculus, Unit 3

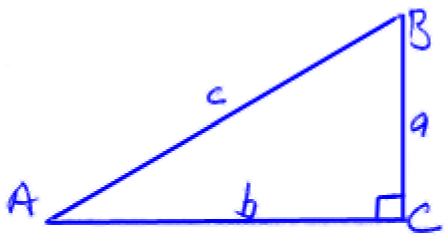
Triangle Solutions

Unit 3: Lesson 01 Abstract solutions of right triangles

There are exactly six basic things to know about a triangle: **3 sides and 3 angles**.

In triangle problems, typically, three things will be given. **"Solving"** the triangle **means determining the other three things**.

Conventional way to name the six parts of a right triangle:



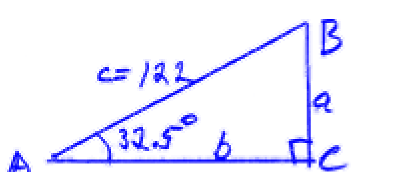
- *a* is opposite angle A
- *b* is opposite angle B
- *c* is opposite angle C

To solve right triangles, make use of the following facts (θ is one of the non-right angles):

- $\sin(\theta) = \text{opp}/\text{hyp}$
- $\cos(\theta) = \text{adj}/\text{hyp}$
- $\tan(\theta) = \text{opp}/\text{adj}$
- Sum of the interior angles is 180° (π radians)

In the following examples, draw and label the right triangle and then solve it using the given information. Notice that when a triangle is a right triangle, this automatically gives one piece of information (one of the angles is 90°).

Example 1: Triangle ABC is a right triangle, $A = 32^\circ 30'$, $c = 122$.



$B + 32.5 + 90 = 180$

$B = \boxed{57.5^\circ}$

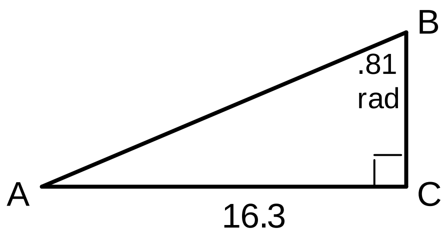
$\sin 32.5 = \frac{a}{122}$

$a = 122 \sin 32.5 = \boxed{65.55}$

$\cos 32.5 = \frac{b}{122}$

$b = 122 \cos(32.5) = \boxed{102.89}$

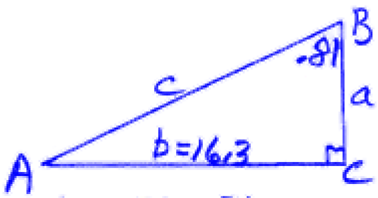
Example 2:



$\sin .81 = \frac{16.3}{c}$

$c \sin .81 = 16.3$

$c = \frac{16.3}{\sin .81} = \boxed{22.5048}$



$A + .81 = \pi/2$

$A = \boxed{.760796 \text{ rad}}$

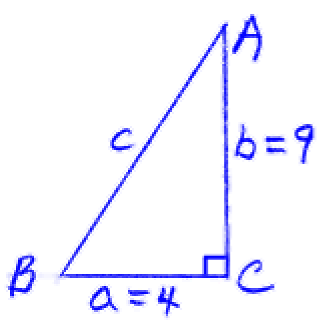
$\tan .81 = \frac{16.3}{a}$

$a \tan .81 = 16.3$

$a = \frac{16.3}{\tan .81} = \boxed{15.51708}$

Occasionally, as in the next example, we are given no angles (besides the right angle); however at least two sides are given. If, for example, we know the side opposite angle θ and the hypotenuse, then on the graphing calculator enter **2nd SIN** and the display will contain **sin^{-1}(** . At this point, enter opp/hyp, close the parenthesis, press ENTER, and the value of the angle θ will appear.

Example 3: Triangle ABC is a right triangle, a = 4, b = 9.



$c^2 = a^2 + b^2$

$c = \sqrt{4^2 + 9^2}$

$c = \boxed{9.848}$

$\sin A = \frac{a}{c} = \frac{4}{9.848} = .40617$

$A = \sin^{-1}(.40617) = \boxed{23.9647^\circ}$

$\sin B = \frac{b}{c} = \frac{9}{9.848} = .91389$

$B = \sin^{-1}(.91389) = \boxed{66.0487^\circ}$

Example 3 above made use of the "inverse sine", **$\sin^{-1}$(opp/hyp)**, to find an angle. Similarly, "inverse cosine" and "inverse tangent" can be used to find angles:

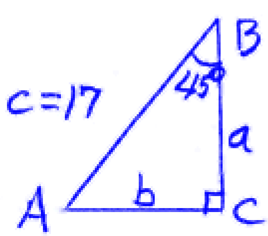
- **$\cos^{-1}$(adj/hyp)**
- **$\tan^{-1}$(opp/adj)**

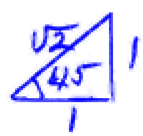
Admittedly, this is a very cursory introduction to inverse functions; however, for the time being, it's a quick way to calculate angles. See **Calculator Appendix U** and a related video for more on inverse trig functions. (In later lessons there will be an in-depth study of inverse trig functions.)

In all of the above examples we have produced **approximate answers** since sine, cosine, and tangent were evaluated on a calculator. These calculations generally produce irrational numbers (decimal places go on forever).

It is possible to produce **exact answers** if the angles are **special angles** (30° , 60°, 90°, 45°, or their radian equivalents).

Example 4: Right triangle ABC, B = 45°, c = 17





$\cos 45 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\sin 45 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\cos 45^\circ = \frac{a}{17}$

$a = 17\cos 45 = \boxed{17\frac{\sqrt{2}}{2}}$ ← exact

$\sin 45^\circ = \frac{b}{17}$

$b = 17\sin 45 = \boxed{17\frac{\sqrt{2}}{2}}$

$A + B = 90$

$A + 45 = 90$

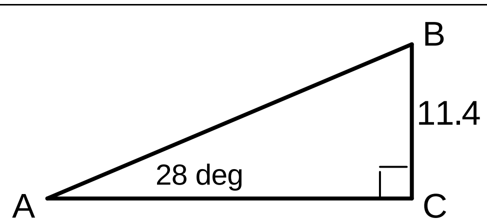
$A = 90 - 45$

$A = \boxed{45^\circ}$

Assignment: In the following problems, assume all triangles are right triangles. Draw and label the right triangle and then solve it using the given information.

1. $A = 46^o$, $b = 19$

2.



3. $A = 20^o$, $B = 70^o$

4. $a = 19$, $c = 24$

5. $B = .98$ radians, $a = 22$

6. $B = 17^{o}\ 45'$, $b = 102$

7. $A = \pi/3$ radians, $c = 23.65$

8. $a = 11$, $b = 13$

In the following two problems, produce **exact** results. All angles are special.

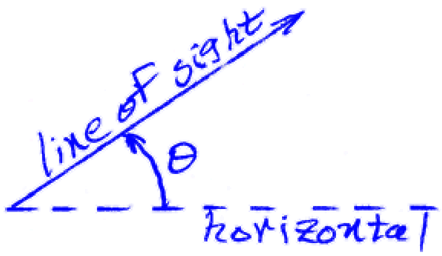
9. $A = 45^{o}$, $b = 17$

10. $B = \pi/6$ radians, $a = 48.07$

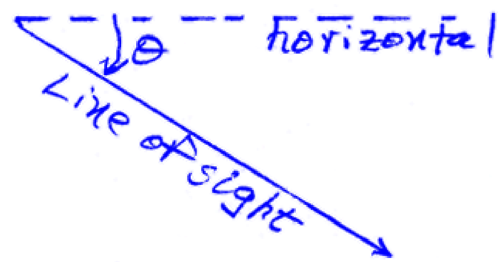
Unit 3: Lesson 02 Right triangle word problems, triangle area

Quite often in solving word problems we are confronted with angles that are not in "standard position."

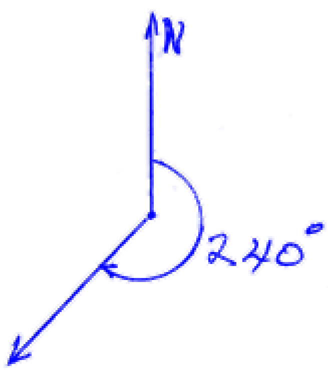
Angle of elevation (measure with respect to a horizontal line):



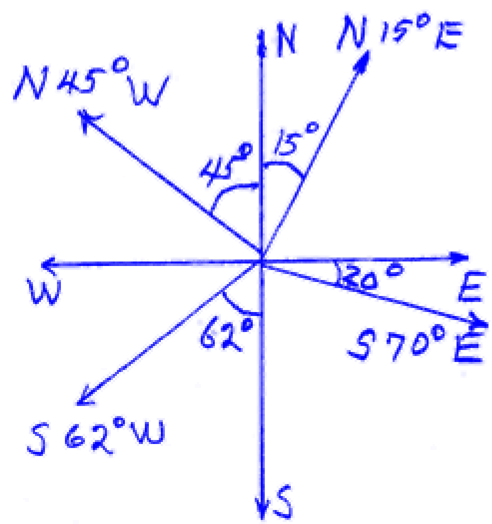
Angle of depression (measure with respect to a horizontal line):



Navigational angle (measure with respect to north, positive direction is clockwise):

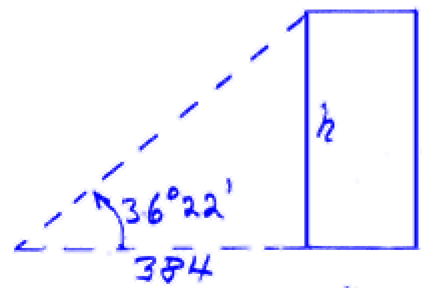


Surveying, bearing angle (the acute angle at which the direction varies to the east or west from the north-south line):



The **area** of a right triangle is given by A = (1/2)(base)(height) which is equivalent to A = .5ab.

Example 1: From a point 384 ft in a horizontal line from the base of a building, the angle of elevation to the top of the building is 36° 22′. How tall is the building?



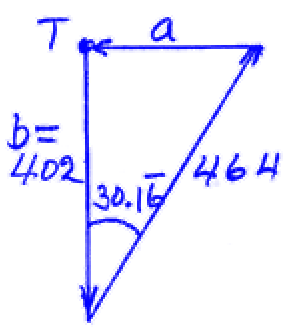
$$\tan 36.3\overline{6} = \frac{h}{384}$$

$$h = 384 \tan(36.3\overline{6})$$

$$h = \boxed{282.7645304}$$

$$\theta = 36°22' = 36 + \frac{22}{60} = 36.3\overline{6}°$$

Example 2: Find the area of a parcel of land whose boundaries are marked as follows: beginning at the old oak tree, thence 402 ft south, thence 464 ft N 30.1666° E, and thence due west back to the old oak tree.



$$\sin 30.1\overline{6} = \frac{a}{464}$$

$$a = 464 \sin(30.1\overline{6})$$

$$a = \boxed{233.1679}$$

$$\text{Area} = \frac{1}{2}ab$$

$$= \frac{1}{2}(233.1679)(402)$$

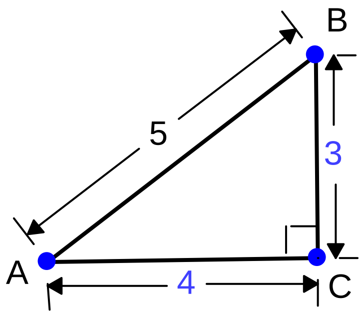
$$= \boxed{46{,}866.74792 \text{ ft}^2}$$

Assignment:

1. Draw an angle of depression of 18^o.	2. Draw an angle of elevation of 71^o.
3. Draw a navigational angle of 280^o.	4. Draw a heading of S 15^o W.

5. An airplane flew from Dweeb City flying at 200^o at a speed of 250 mph for 2 hours and reached Nerdtown. From that point the plane turned and flew due east and landed at Geekville which just happened to be due south of Dweeb City. What is the distance from Nerdtown to Geekville?

6. Find the area of this triangle.



7. Find the area of right triangle ABC where a = 79 and c = 103.

8. My house is 500 yards south of your house, and this distance subtends an angle of 20^{o} at a point P that is due west of your house. How far is point P from my house?

9. A certain piece of land is in the shape of a right triangle. The longest side is 842 meters and bears S 36^{o} W. How many meters of fence would enclose this tract if one of the sides runs east-west?

10. A cell-phone tower is 380 ft tall. An observer is how far from the base of this tower if he and the tower are on level ground and the angle of elevation from the observer to the top of the tower is 30^{o} 15'?

11. The elevation above sea level at the entrance to a mine is 1600 ft. The mine shaft descends in a straight line for 300 ft at an angle of depression of 24^{o}. Find the elevation of the bottom of the mine shaft above sea level.

12. A piece of land slopes at an angle of 3^{o} and runs for 280 ft in the direction of the slope. In order to level the land, a retaining wall is to be built at the lower end of the property so that fill-dirt can level the property. How high must the wall be?

13. An airplane flew at 20^{o} for 5 hrs at 180 mph. It then turned and headed due west and landed at a point directly north of the starting point. After refueling, the plane then flew due south back to the starting point. What is the area in sq miles of the triangular pattern flown by the airplane?

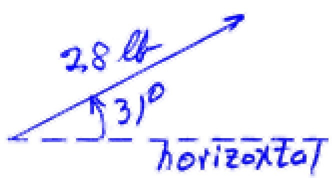
Unit 3: Lesson 03 Vectors

Quantities that have both **magnitude**(size) and **direction** are known as **vectors**.

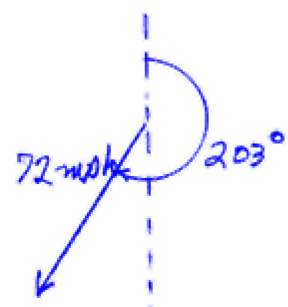
Examples of vector quantities are forces, velocities, accelerations, and displacements.

Vectors are **represented graphically with an arrow** (a directed line-segment) where the length of the arrow is proportional to the magnitude of the vector. Of course, the direction of the arrow represents the direction of the vector.

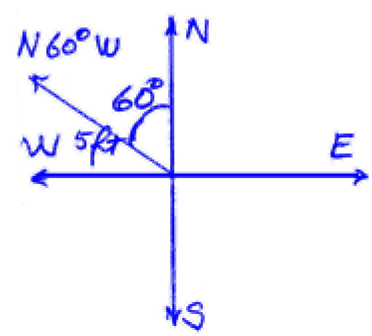
Example 1: Force



Example 2: Velocity

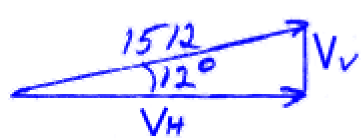


Example 3: Displacement



Vectors can be resolved into two mutually perpendicular components: usually, horizontal and vertical components. If our vector is a force represented by the symbol **F**, then the horizontal and vertical components are called F_H and F_V (or equivalently F_X and F_Y).

Example 4: The velocity of a bullet is 1512 ft/sec and inclines upward at 12^o. Resolve this velocity (call it **V**) into its horizontal and vertical components.



$\sin 12 = V_V/1512$; $V_V = 1512 \sin 12$

$V_V = \boxed{314.36 \text{ ft/sec}}$

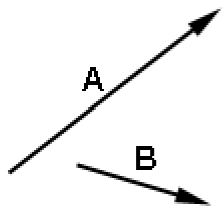
$\cos 12 = V_H/1512$; $V_H = 1512 \cos 12$

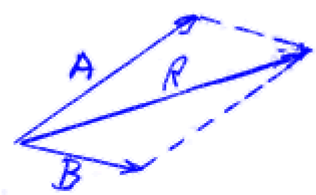
$V_H = \boxed{1478.96 \text{ ft/sec}}$

Addition of vectors:
The sum of vectors is called the **resultant**. Vectors can be added by one of three methods:

Parallelogram method (for adding two vectors): Move the two vectors (keeping them parallel to their original directions) together tail-to-tail and complete the parallelogram. The **diagonal** from the two joined tails to the opposite corner **is the resultant** (the sum).

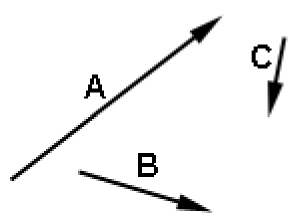
Example 5: Using the parallelogram method, graphically show the resultant **R** when adding vectors **A** and **B**.

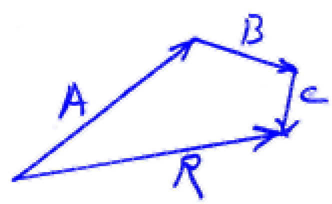




Head-to-tail method (for adding any number of vectors): Move the vectors (keeping them parallel to their original directions) so that they daisy-chain together in a head-to-tail fashion. The resultant (the sum) is drawn from the tail of the first vector in the chain to the head of the last.

Example 6: Using the head-to-tail method, graphically show the resultant **R** when adding vectors **A**, **B**, and **C**.

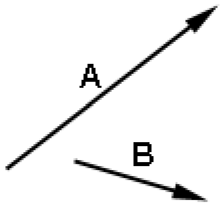




Component method: Break each vector up into its two perpendicular components. Add all the horizontal components and call the sum R_H (or R_X). Add all the vertical components and call the sum R_V (or R_Y).

Create the final resultant by adding these two vector components.

Example 7: Using the component method, add vectors **A** (angle of elevation 40^o) and **B** (angle of depression 19^o). The magnitude of **A** is 107 and the magnitude of **B** is 51.

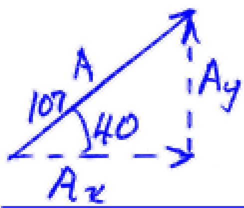


$\sin 19 = B_y/51$ minus because down

$\boxed{B_y = -16.604}$

$\cos 19 = B_x/51$

$\boxed{B_x = 48.22}$



$\sin 40 = A_y/107$; $A_y = 68.78$

$\cos 40 = A_x/107$; $A_x = 81.97$

Resultant

$R_x = A_x + B_x = 81.97 + 48.22$

$= 130.19$

$R_y = A_y + B_y = 68.78 - 16.604$

$= 52.176$

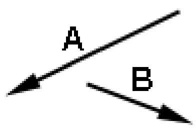
$R = \sqrt{130.19^2 + 52.176^2}$

$R = \boxed{140.26}$

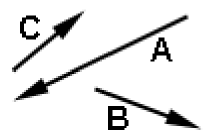
$\theta = \tan^{-1}\left(\frac{52.176}{130.19}\right) = \boxed{21.84^\circ}$

Assignment:

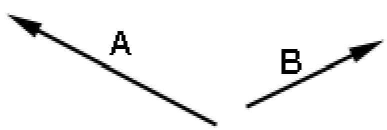
1. Add these vectors graphically using the parallelogram method.



2. Add these vectors graphically using the head-to-tail method.



3. Add vectors **A** and **B** using the component method. Vector **A** has a magnitude of 540 and an angle of elevation of 23^{o}. Vector **B** has a magnitude of 312 and an angle of elevation of 20^{o}.



4. An object is moving with a speed of 180 m/sec in an x-y coordinate system. Its direction is that described by a 137° angle in standard position. What are the *X* and *Y* components of the velocity of the object?

5. Two forces, $\mathbf{F_1}$ and $\mathbf{F_2}$, are pulling on the same object. $\mathbf{F_1}$ is a force of 300 lb with a bearing of S 40° E. $\mathbf{F_2}$ is a force of 150 lb with a bearing of N 50° E. Use the parallelogram method to find the single force (the resultant) that equivalently replaces these two forces. Give both the magnitude and bearing of the resultant.

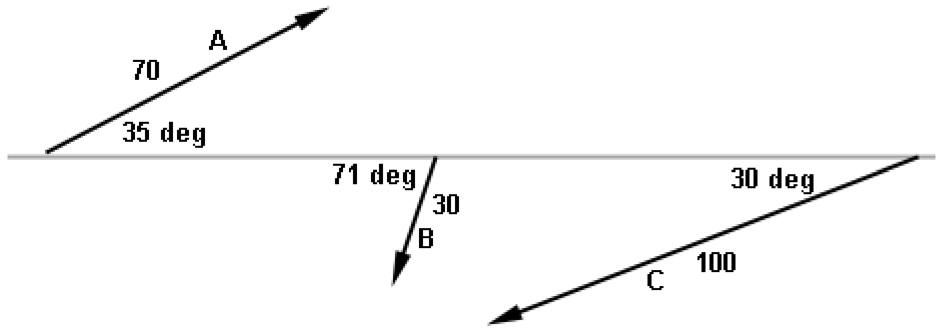
6. Draw the displacement vector **D** that has a magnitude of 136 and bearing N 18° W.	7. Draw the velocity vector **V** of a boat with a heading of 350° and speed 37 mph.

8. A football is thrown due south at 5 m/sec from a car that is traveling east at 30 m/sec. Find the speed of the ball relative to the ground and its direction of travel.

---

9. An airplane is headed due north with a speed of 205 mph. The wind is blowing from the east at a speed of 38 mph. What is the navigational heading and speed of the plane’s shadow on the ground?

10. Using the component method, find the magnitude and direction(as an angle in standard position) of the resultant after adding these three displacement vectors.

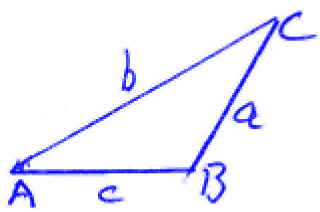


Unit 3: Lesson 04 Sine Law; more triangle area formulas

Until this point in our studies we have been restricted to solutions of only right triangles. No longer.

The **sine law** applies to **any type triangle**, not just right triangles.

Each of the following proportions is an expression of the sine law. Notice that each ratio is made of a pair of items that are **opposite each other** in the triangle.



$$\frac{\sin C}{c} = \frac{\sin B}{b} = \frac{\sin A}{a}$$

For a derivation of the sine law see **Enrichment Topic K**.

By dropping a perpendicular from a vertex to the line described by the opposite side, and then using right triangle identities and the sine law, it is possible to produce the following two important triangle area formulas. It is conventional to use **K** for the area of a triangle.

$$K = \frac{1}{2}\frac{b^2 \sin A \ \sin C}{\sin B}$$

Since the names of sides and angles within an oblique triangle are interchangeable, this area formula should be learned as follows:

The area of a triangle is equal to one-half the product of the square of any side and the sines of the adjacent angles divided by the sine of the opposite angle.

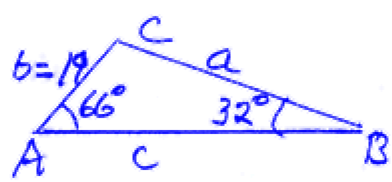
Another area formula:

$$K = \frac{1}{2} a\, b \sin(C)$$

Since the names of sides and angles within an oblique triangle are interchangeable, this area formula should be learned as follows:

The area of a triangle is equal to one-half the product of any two sides times the sine of their included angle.

Example 1: A = 66°, B = 32°, b = 19: Solve this triangle and find its area.



$C+66+32=180 \rightarrow C = \boxed{82°}$

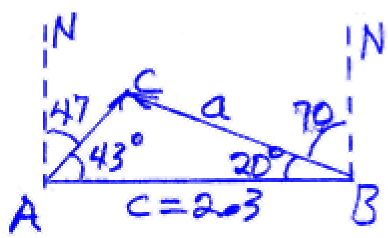
$\frac{\sin 32}{19} = \frac{\sin 66}{a} \rightarrow a = \boxed{32.75}$

$\frac{\sin 32}{19} = \frac{\sin 82}{c} \rightarrow c = \boxed{35.51}$

$K = \frac{1}{2} ab \sin C$

$= \frac{1}{2}(32.75)(19)\sin 82° = \boxed{308.097}$

Example 2: Two observers are stationed on an east-west line and are 2.3 miles apart. Observer A sees the steeple of the old church with a bearing of N 47° E. Observer B reports a bearing of N 70° W. How far is the church from B?



$C + 20 + 43 = 180$

$C = 117$

$\frac{\sin 43}{a} = \frac{\sin 117}{2.3}$

$a \sin 117 = 2.3 \sin 43$

$a = \frac{2.3 \sin 43}{\sin 117}$

$a = \boxed{1.76 \text{ mi}}$

Assignment: In problems 1-3, solve the triangle and then find its area. Draw and fully label the triangle.

1. $A = 16^{o}$, $B = 57^{o}$, $a = 156.9$

---

2. $A = 46^{o}$, $C = 68^{o}$, $b = 11.04$

---

3. $B = 21^{o}$, $C = 61^{o}$, $c = 460$

4. From a bridge above a river an observer looks along an angle of depression of 77^o to a rock on the bank (directly under the bridge). Walking forward 40 ft, she determines a new angle of depression of 88^o to the same rock. How far is the rock from her second position?

5. A surveyor at the top of a pyramid in Egypt, whose surface makes a 52^o angle with respect to the horizontal, finds the angle of depression to the bottom of a building on the horizontal plain below is 35^o. A laser range finder shows the distance from his position to the bottom of the building is 1850 ft. How far is the building from the base of the pyramid?

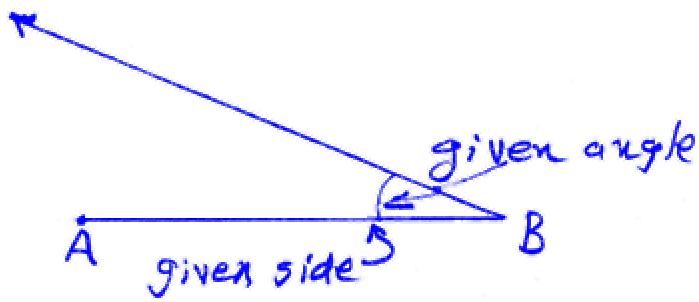
6. An observer in Dweeb City sights a UFO at a bearing of N 45^o E. Simultaneously, an observer in Nerdville sights the same UFO with a bearing of N 60^o W. How far is the UFO from Nerdville if Nerdville is 3.6 miles N 76^o E of Dweeb City?

Unit 3: Lesson 05 Ambiguous case of the sine law

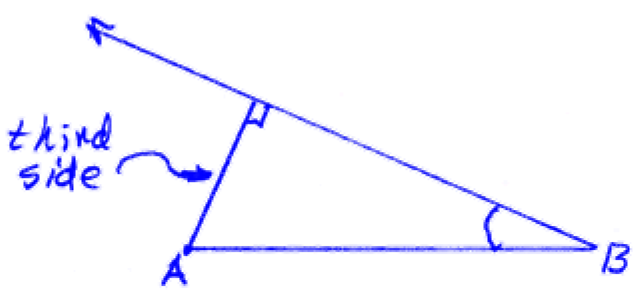
In the previous lesson the astute observer might have noticed that the three pieces of information initially given about a triangle never consisted of **two sides and a non-included angle**.

This is known as the **ambiguous case** since there are three possibilities for the solution.

First, consider being given a side (a line segment) and an angle at one end of that segment as shown here:

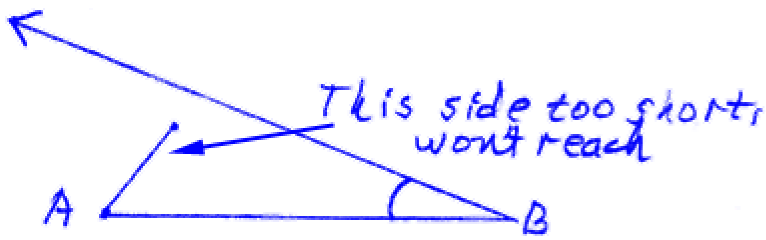


Case 1: Consider a third side that is opposite the angle (**now we have two sides and a non-included angle**) that is just barely long enough to reach the other side of the angle in a **perpendicular** fashion.



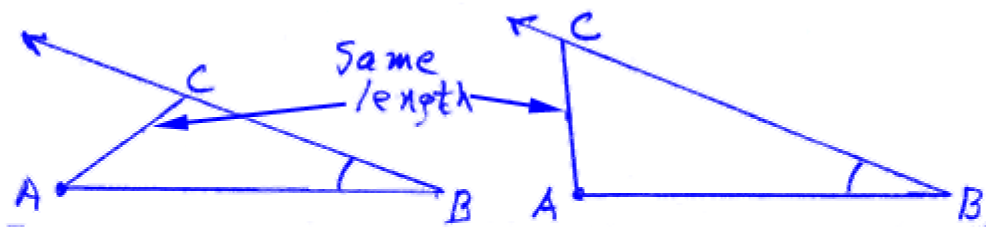
Mathematically, you will be alerted to this when the sine calculation of an angle yields a value of 1 (remember, sin 90^{o} = 1).

Case 2: Consider a third side that is opposite the angle and is **too short** to reach the other side of the angle.



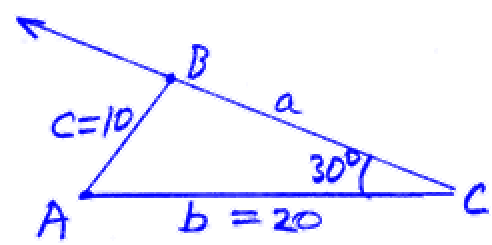
Mathematically, you will be alerted to this unpleasant possibility when the sine or cosine calculation of an angle yields an impossible value (outside the acceptable range, -1 ≤ value ≤ 1).

Case 3: Consider a third side (that is opposite the angle) that can actually touch the other side of the angle in **two places**.



Two different triangles are possible, and it must be decided from the physical situation represented by the problem if both solutions are acceptable or if one must be rejected.

Example 1 (representing case 1): Solve triangle ABC where b = 20, c = 10, C = 30°.



$$\frac{\sin 30^\circ}{10} = \frac{\sin B}{20}$$

$$\sin B = 1$$

$$B = \sin^{-1}(1) = \boxed{90^\circ}$$

$$90 + 30 + A = 180$$

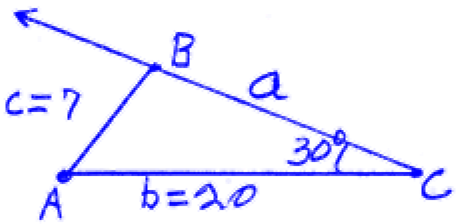
$$A = \boxed{60^\circ}$$

$$\frac{\sin 30}{10} = \frac{\sin 60}{a}$$

$$a \sin 30 = 10 \sin 60$$

$$a = \boxed{17.3205}$$

Example 2 (representing case 2): Solve triangle ABC where b = 20, c = 7, C = 30°.



$$\frac{\sin 30}{7} = \frac{\sin B}{20}$$

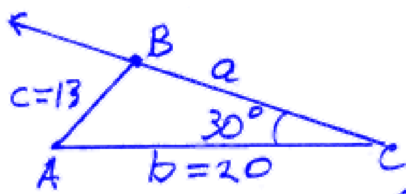
$$\sin B = (20 \sin 30)/7$$

$$\sin B = 1.428$$

Illegal, outside the range of $-1 \longleftrightarrow 1$.

$\boxed{\text{Triangle is not possible; side c is too short!}}$

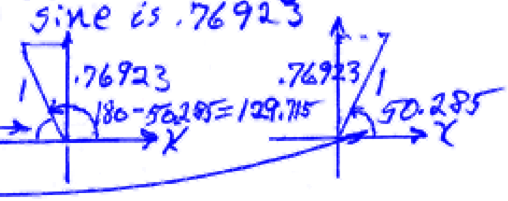
Example 3 (representing case 3): Solve triangle ABC where b = 20, c = 13, C = 30°.

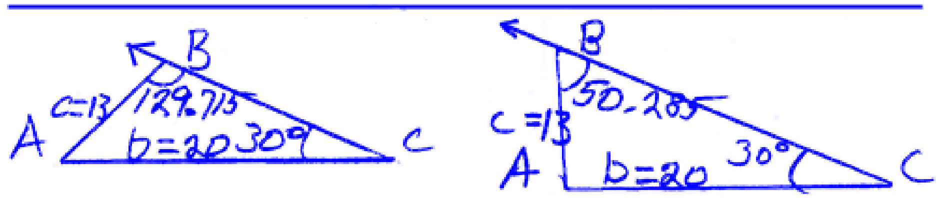


$$\frac{\sin 30}{13} = \frac{\sin B}{20};\ \sin B = .76923$$

$$B = \sin^{-1}(.76923) = 50.285°$$

There are 2 angles whose sine is .76923





B can be either of two values

$B = \boxed{129.715}$	$B = \boxed{50.285}$
$A + 30 + 129.715 = 180$	$A + 30 + 50.285 = 180$
$A = \boxed{20.285°}$	$A = \boxed{99.715°}$
$\frac{\sin 30}{13} = \frac{\sin 20.285}{a}$	$\frac{\sin 30}{13} = \frac{\sin 99.715}{a}$
$a = \boxed{9.014}$	$a = \boxed{25.627}$

Assignment:

1. Solve triangle ABC where $c = 100$, $B = 62^o$, $b = 91$

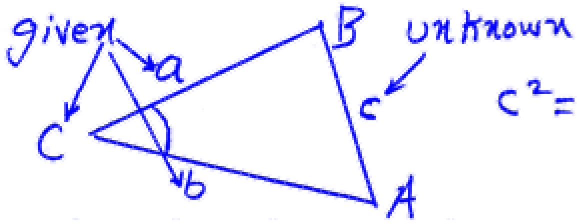
2. Solve triangle ABC where $c = 100$, $B = 62^o$, $b = 71$

3. Solve triangle ABC where $c = 100$, $B = 59.3347^o$, $b = 86.01613136$

4. A pilot flies at a heading of 138^o from A to B and then at 235^o from B to C. If A is 600 miles from B and A is 788 miles from C, how far is it from C to B?

Unit 3: Lesson 06 Cosine Law

In the previous lessons a problem was never given in which **two sides and their included angle** was the initial information given. Application of the sine law always results in a single equation in two variables: all progress stops. **To solve such a problem the cosine law is needed**:



$c^2 = a^2 + b^2 - 2ab\cos C$

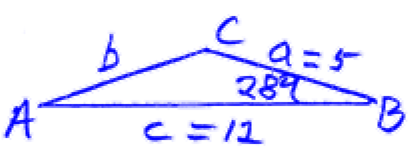
cosine law

Since the angles and corresponding opposite sides could have been labeled differently, the **cosine law** should be learned as follows:

> ***The square of a side equals the sum of the squares of the other two sides minus twice the product of the others sides and the cosine of the angle between them.***

See **Enrichment Topic L** for a derivation of the cosine law.

Example 1: Solve the triangle ABC where a = 5, c = 12, B = 28°.



$b = \sqrt{12^2 + 5^2 - 2(12)5\cos 28}$

$b = \boxed{7.94}$

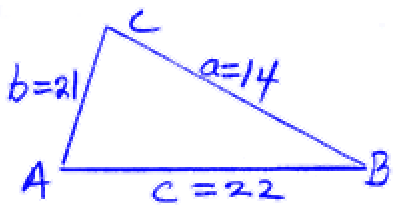
We now work with the ambiguous case of the sine law and we have two ways to go as shown below. We could work with 5 or 12. Always choose the smaller so as to produce a simpler acute angle.

$\frac{\sin 28}{7.94} = \frac{\sin A}{5}$ or $\cancel{\frac{\sin 28}{7.94} = \frac{\sin C}{12}}$

$A = \boxed{17.20°}$ $\quad A + B + C = 180$

$C = \boxed{134.8°}$

Example 2: Find the angles of the triangle whose sides are a = 14, b= 21, and c = 22.



$$a^2+c^2-2ac\cos B=b^2$$

$$\cos B=\frac{b^2-a^2-c^2}{-2ac}$$

$$\cos B=.388$$

$$B=\cos^{-1}(.388)=\boxed{67.17^\circ}$$

$$\frac{\sin 67.17}{21}=\frac{\sin A}{14}$$

Notice we choose the smaller of 14 and 22

$$\sin A=.614$$

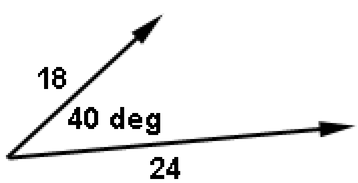
$$A=\sin^{-1}(.614)=\boxed{37.91^\circ}$$

$$A+B+C=180$$

$$37.91+67.17+C=180$$

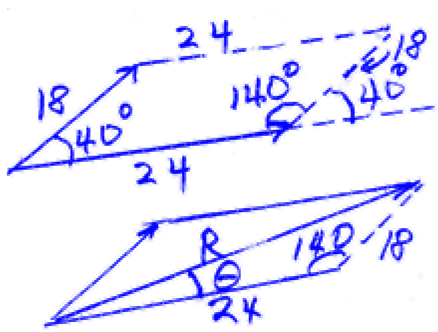
$$C=\boxed{74.92^\circ}$$

Example 3: Use the parallelogram method to add these vectors. Find both magnitude and direction of the resultant.



$$R=\sqrt{24^2+18^2-2(24)(18)\cos 140}$$

$$R=\boxed{39.52}$$



Direction is given by θ, angle with side 24.

$$\frac{\sin 140}{39.52}=\frac{\sin\theta}{18}$$

Notice we choose the smaller of 18 and 24.

$$\sin\theta=.292767$$

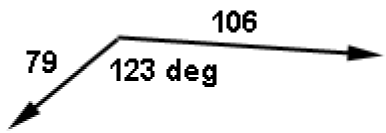
$$\theta=\sin^{-1}(.292767)=\boxed{17.02^\circ}$$

Assignment:

1. Find the angles of a triangle whose sides are: a = 20, b= 28, c = 21

2. Solve the triangle ABC where a = 200, b = 300, C = 130^o.

3. Use the parallelogram method to add these vectors. Find both magnitude and direction of the resultant.



4. Two forces of 250 lbs and 600 lbs are acting at the same point and make an angle of 65° with each other. Find the magnitude of the resultant and its direction (the angle it makes with the 600 lb force).

5. Orville and Wilbur leave the airport at the same time. Wilbur files S 37^o E at 65 mph while Orville flies N 59^o W at 102 mph. How far apart are they after 4 hours?

Unit 3: Cumulative Review

1. Solve $4x^2 - 2x + 1 = 0$ with the quadratic formula.

2. Solve $3x^2 + x - 6 = 0$ by completing the square.

3. Solve $x^2 + 2x - 35 = 0$ by factoring.

4. Multiply $(2x^2 - 3y)(x^2 - 7y)$

5. Define sin, cos, & tan in terms of opp, adj, and hyp as these terms apply to a right triangle.

6. Define sin, cos, & tan in terms of x, y, and r.

7. Simplify $\dfrac{a+\frac{3}{by}}{\frac{b}{y^2}+6}$

8. Convert 3π/7 radians to degrees.

9. Convert 26.12^{o} to radians.

10. An angle, θ, subtends an arc of 1.2 meters and has a radius of 14 meters. What is the value of the θ in radians?

11. Find the equation of the line (in slope-intercept form) that passes through (4, –1) and is perpendicular to the line given by $x/3 + y/7 = 1$.

12. Draw a 30-60-90 triangle and label the standard lengths of the sides.	13. Draw a 45-45-90 triangle and label the standard lengths of the sides.

14. Solve this system of equations: $3x - 6y = 5$ and $x + 3y = 5$

Unit 3: Review

1. Solve right triangle ABC where a = 3, and B= 19^o.

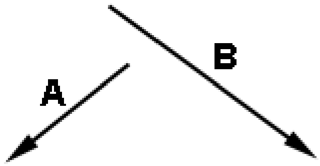
2. Find the area of right triangle ABC where a = 4.7 and c = 18.

3. Two planes leave an airport at noon. How far apart are they at 2:00 PM when one flies with a speed of 180 mph at a heading of 340^o and the other with a speed of 200 mph at a heading 250^o?

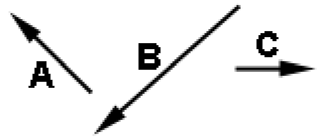
4. Write a formula for the cosine law in terms of the sides and angles of a triangle (a, b, c, A, B, C).

5. Write a formula for the sine law in terms of the sides and angles of a triangle (a, b, c, A, B, C).

6. Graphically show how to add these two vectors with the parallelogram method.



7. Graphically show how to add these three vectors with the head-to-tail method.



8. A pilot is flying at 183 mph. The direction in which the plane is headed makes an angle of 38^o with the direction of the wind which is blowing at 46 mph. Use the parallelogram method to determine the ground speed of the plane. How far off-course is the plane from the direction in which it tries to head?

9. What information about a triangle constitutes the "ambiguous case" of the sine law?

10. Vector A consists of two components, $A_X = 5$ and $A_Y = -11$. Vector B has magnitude 8 and has a direction described by an angle of 120^o in standard position. Find the magnitude of the sum of vectors A and B using the component method.

11. Find the exact values of the cosecant and tangent of 210^o.

12. Solve triangle ABC where $A = 20^o$, $B = 32^o$, $a = 12$.

13. Solve triangle ABC where $A = 56^o$, $b = 9$, $a = 21$.

14. What is the component of the vector of magnitude 86 and bearing S 41^o E in the east direction?

Pre Calculus, Unit 4

Trig identities

Unit 4: Lesson 01 Reciprocal & Pythagorean identities, trig simplifications

Reciprocal identities:
Recall that sin θ = y/r and csc θ = r/ y. This lets us write:

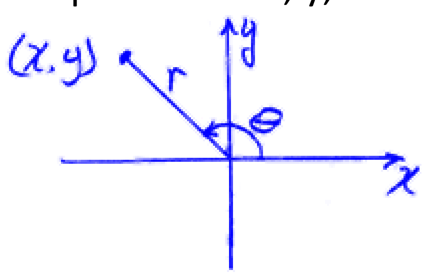
$$\sin\theta = \frac{1}{\csc\theta} \quad \text{or} \quad \sin\theta\csc\theta = 1$$

Similarly, we can write the other two reciprocal identities:

$$\cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\text{or } \cos\theta\sec\theta = 1 \qquad \text{or } \tan\theta\cot\theta = 1$$

Pythagorean identities:
Consider the end point (x, y), of a rotating unit (radius 1) vector and the relationship between x, y, and r (regardless of the quadrant):



$$x^2 + y^2 = r^2$$

Derivation 1: Divide both sides by r^2.

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2}$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1 \longrightarrow \boxed{\cos^2\theta + \sin^2\theta = 1}$$

Notice that **$\sin^2\theta$ really means $(\sin\theta)^2$**. Similarly, $\cos^2\theta$ really means $(\cos\theta)^2$, etc.

Derivation 2: Divide both sides by x^2.

$$\frac{x^2}{x^2} + \frac{y^2}{x^2} = \frac{r^2}{x^2}$$

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2 \longrightarrow \boxed{1 + \tan^2\theta = \sec^2\theta}$$

Derivation 3: Divide both sides by y^2.

$$\frac{x^2}{y^2} + \frac{y^2}{y^2} = \frac{r^2}{y^2}$$

$$\left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2 \longrightarrow \boxed{\cot^2\theta + 1 = \csc^2\theta}$$

Summary: These identities must be memorized!

Reciprocal Identities:	**Pythagorean Identities:**
$\sin\theta = \dfrac{1}{\csc\theta}$	$\sin^2\theta + \cos^2\theta = 1$
$\cos\theta = \dfrac{1}{\sec\theta}$	$1 + \tan^2\theta = \sec^2\theta$
$\tan\theta = \dfrac{1}{\cot\theta}$	$1 + \cot^2\theta = \csc^2\theta$

We need just two more identities involving tan and cot.

Derivation 4: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$\tan\theta = \frac{y}{x}\,\frac{1/r}{1/r} = \frac{y/r}{x/r} = \boxed{\frac{\sin\theta}{\cos\theta}}$$

Derivation 5: $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

$$\cot\theta = \frac{x}{y}\,\frac{1/r}{1/r} = \frac{x/r}{y/r} = \boxed{\frac{\cos\theta}{\sin\theta}}$$

Armed with these identities, it is possible to simplify trig expressions.

Example 1: Simplify sec(θ)/csc(θ)

$$\frac{\sec\theta}{\csc\theta} = \frac{1/\cos\theta}{1/\sin\theta} = \frac{1}{\cos\theta}\,\frac{\sin\theta}{1} = \boxed{\tan\theta}$$

Example 2: Simplify $\dfrac{\tan\theta(1+\cot^2\theta)}{1+\tan^2\theta}$

$$\frac{\tan\theta(1+\cot^2\theta)}{1+\tan^2\theta}=\frac{\tan\theta\csc^2\theta}{\sec^2\theta}$$

$$=\frac{\dfrac{\cancel{\sin\theta}}{\cos\theta}\,\dfrac{1}{\sin^{\cancel{2}}\theta}}{1/\cos^2\theta}=\frac{\dfrac{1}{\sin\theta\cos\theta}}{\dfrac{1}{\cos^2\theta}}$$

$$=\frac{1}{\sin\theta\cancel{\cos\theta}}\,\frac{\cos^{\cancel{2}}\theta}{1}=\frac{\cos\theta}{\sin\theta}$$

$$=\boxed{\cot\theta}$$

Assignment: Simplify these trig expressions so as to produce the answers in **bold print**.

1. cos(A) tan(A) **sin(A)**

2. sin(A) cot(A) **cos(A)**

3. sec(A) – cos(A) **tan(A) sin(A)**

4. csc A – sin A **cotA cosA**

5. $(1 + \sin A)(1 - \sin A)$ **$\cos^2 A$**

6. $(\sec A + 1)(\sec A - 1)$ **$\tan^2 A$**

7. $\frac{1-\sin^2 B}{\cot B}$ **$\sin B \cos B$**

8. $\frac{\cos A}{\csc^2 A - 1}$ **$\tan A \sin A$**

9. $\dfrac{1}{\sec A - \tan A} + \dfrac{1}{\sec A + \tan A}$ **2sec A**

Unit 4: Lesson 02 Trig Proofs

In the last lesson we started with a trig expression and then simplified. This lesson is very similar; however, we start with an equation involving a trig expression and show that it is true.

What is an identity?

An **identity** is an equation that is true no matter what the value of the variable(s) is (are).

Example of an identity:

$x^2-9=(x-3)(x+3)$
true for any x

Example of a non-identity (a **conditional equation**):

$3x+2=-4$
only true for $x=-2$

To prove an identity work with either or both sides until the result is **something equals itself**.

Example 1: Prove $\dfrac{\cos\theta(\cos\theta-\sin\theta)}{(1-2\sin\theta\cos\theta)}=\dfrac{\cos\theta}{\cos\theta-\sin\theta}$

cross multiply & get

$\cancel{\cos\theta}(\cos^2\theta-2\cos\theta\sin\theta+\sin^2\theta)=\cancel{\cos\theta}(1-2\sin\theta\cos\theta)$

$\underbrace{\cos^2\theta+\sin^2\theta}-2\cos\theta\sin\theta=1-2\sin\theta\cos\theta$

$1-2\sin\theta\cos\theta \overset{\checkmark}{=} 1-2\sin\theta\cos\theta$

Example 2: Prove $\sin B(\csc B - \sin B) = \cos^2 B$

$$\sin B(\csc B - \sin B) = \cos^2 B$$

$$\underbrace{\sin B \csc B} - \sin^2 B = \cos^2 B$$

$$1 - \sin^2 B = \cos^2 B$$

$$1 = \underbrace{\sin^2 B + \cos^2 B}$$

$$1 \overset{\checkmark}{=} 1$$

Assignment: Prove the following identities:

1. $\frac{\tan\theta}{1+\sec\theta} - \frac{\tan\theta}{1-\sec\theta} = \frac{2}{\sin\theta}$

2. $\tan\theta(\cot\theta + \tan\theta) = \sec^2\theta$

3. $(1 + \cos B)(1 - \cos B) = \sin^2 B$

4. $\cos^2B(\sec^2B + \tan^2B + 1) = 2$

5. $\frac{\csc A - \sin A}{\cot A} - \frac{\cot A}{\csc A} = 0$

6. $\sin\theta \csc\theta = \frac{\sin\theta}{\csc\theta} + \frac{\cos\theta}{\sec\theta}$

7. $\frac{\sec^2 A - \tan^2 A + \tan A}{\sec A}$ = sinA + cosA

---

8. $\frac{\sin A}{\csc A - 1} = \frac{1 + \sin A}{\cot^2 A}$

---

9. $\frac{1}{1 + \sin A} = \frac{1 - \sin A}{\cos^2 A}$

10. $(\sec\phi + 1)(\sec\phi - 1) = \tan^2\phi$

Unit 4: Lesson 03 Cosine composite angle Identities

Without proof we submit the identities for the cosine of the sum and difference of angles:

cos(A – B) = cosA cosB + sinA sin B
cos(A + B) = cosA cosB – sinA sin B

These are often written compactly together as:

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

Example 1: Find cos(A + B) where sinA = –3/5, π ≤ A ≤ 3π/2; cosB = 1/5, 3π/2 ≤ B ≤ 2π.

$$\sin^2 A + \cos^2 A = 1$$
$$(-3/5)^2 + \cos^2 A = 1$$
$$\cos A = \pm\sqrt{1-(-3/5)^2}$$
$$= -4/5 \quad \text{neg, 3rd Quad}$$

$$\sin^2 B + \cos^2 B = 1$$
$$\sin^2 B + \left(\tfrac{1}{5}\right)^2 = 1$$
$$\sin B = \pm\sqrt{1-\left(\tfrac{1}{5}\right)^2}$$
$$\sin B = -\frac{2\sqrt{6}}{5} \quad \text{Neg, 4th Quad}$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$= -\frac{4}{5}\cdot\frac{1}{5} - \left(-\frac{3}{5}\right)\left(\frac{2\sqrt{6}}{5}\right) = \boxed{\frac{-4-6\sqrt{6}}{25}}$$

The identity for the cosine of twice an angle, cos 2A= $\cos^2$A – $\sin^2$A, is easily proved:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
let B=A $\quad \cos(A+A) = \cos A \cos A - \sin A \sin A$
$$\boxed{\cos(2A) = \cos^2 A - \sin^2 A}$$

Making substitutions from $\sin^2$A + $\cos^2$A = 1, there are three forms of the cosine of 2A:

cos 2A = $\cos^2$A – $\sin^2$A
cos 2A = 1 – 2$\sin^2$A
cos 2A = 2$\cos^2$A – 1

Example 2: Find cos2θ when sinθ = –15/17, π ≤ θ ≤ 3π/2.

$$\cos 2\theta = 1 - 2\sin^2\theta$$
$$= 1 - 2\left(-\frac{15}{17}\right)^2 = 1 - \frac{450}{289}$$
$$= \frac{289}{289} - \frac{450}{289} = \boxed{\frac{-161}{289}}$$

The identity for the cosine of a half-angle is easily derived from cos2θ = 2cos^{2}θ –1:

Let $2\theta = A$
$\theta = \frac{A}{2}$

$$2\cos^2\theta - 1 = \cos 2\theta$$
$$2\cos^2\frac{A}{2} - 1 = \cos A$$
$$\cos^2\frac{A}{2} = \frac{\cos A + 1}{2}$$
$$\boxed{\cos\frac{A}{2} = \pm\sqrt{\frac{\cos A + 1}{2}}}$$

Example 3: Find cos(A/2) when sinA = 4/5, π/2 ≤ A ≤ π.

$\sin^2 A + \cos^2 A = 1$
$\left(\frac{4}{5}\right)^2 + \cos^2 A = 1$
$\cos A = \pm\sqrt{1 - \frac{16}{25}}$
$\cos A = -\frac{3}{5}$; Neg, 2nd quad

$\cos(A/2) = \pm\sqrt{\frac{1+\cos A}{2}}$
$= \pm\sqrt{\frac{1 - 3/5}{2}} = \pm\sqrt{\frac{1}{5}}$
$= \boxed{\frac{\sqrt{5}}{5}}$; pos, 1st quad

1st Quad

Even and odd properties

cos(x) is an even function, so **cos(–x) = cos(x)** because all even functions obey f(–x) = f(x).

sin(x) and tan(x) are odd functions, so **sin(–x) = –sin(x)** and **tan(–x) = –tan(x)** because all odd functions obey f(–x) = –f(x).

Assignment:

1. Find $\cos(A - B)$ where $\sin A = -4/5$, $\pi \le A \le 3\pi/2$; $\cos B = 3/4$, $3\pi/2 \le B \le 2\pi$.

2. Find $\cos 2\theta$ when $\cos\theta = 1/3$, $0 \le \theta \le \pi/2$.

3. Find $\cos(A/2)$ when $\cos A = -4/5$, $\pi/2 \le A \le \pi$.

4. Simplify $-\sin(-3x)$

5. Simplify $-\cos(-5\phi)$

6. Use the exact function values of 30^o, 45^o, and/or 60^o to find $\cos(105^o)$.

7. Use the exact function values of 30^o, 45^o, and/or 60^o to find $\cos(22^o\ 30')$.

8. Use the exact function values of 30^o, 45^o, and/or 60^o to find $\cos(120^o)$.

9. Consider evaluating $\cos(A + B)$. If A and B are unknown but the sines and cosines of these angles are known, what could be used to evaluate this composite function?

Prove the following identities:

10. $\cos 3A \cos 5A - \sin 3A \sin 5A = \cos 8A$

11. $(1 - 2\sin^2\phi)^2 = 1 - \sin^2 2\phi$

12. $\cos 3A \cos A - \sin 3A \sin A = 2\cos^2 2A - 1$

Unit 4: Lesson 04 Sine composite angle identities

Without proof we submit the identities for the sine of the sum and difference of two angles:

$$\sin(A - B) = \sin A \cos B - \sin B \cos A$$
$$\sin(A + B) = \sin A \cos B + \sin B \cos A$$

These are often written compactly together as:

$$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$$

Example 1: Find sin(A – B) where sinA = –3/5, π ≤ A ≤ 3π/2; cosB = 1/5, 3π/2 ≤ B ≤ 2π.

$$\sin^2 A + \cos^2 A = 1$$
$$\left(-\tfrac{3}{5}\right)^2 + \cos^2 A = 1$$
$$\cos A = \pm\sqrt{1 - \tfrac{9}{25}}$$
$$\cos A = -\tfrac{4}{5}\text{; neg 3rd Quad}$$

$$\sin^2 B + \cos^2 B = 1$$
$$\sin^2 B + \left(\tfrac{1}{5}\right)^2 = 1$$
$$\sin B = -\tfrac{2\sqrt{6}}{5}$$
neg, 4th Quad

$$\sin(A-B) = \sin A \cos B - \sin B \cos A$$
$$= -\tfrac{3}{5}\left(\tfrac{1}{5}\right) - \tfrac{-2\sqrt{6}}{5}\left(-\tfrac{4}{5}\right)$$
$$= -\tfrac{3}{25} - \tfrac{8\sqrt{6}}{25}$$
$$= \boxed{\tfrac{-3 - 8\sqrt{6}}{25}}$$

The identity for the sine of twice an angle, **sin 2A = 2sinA cosA**, is easily proved:

$$\sin(A+B) = \sin A \cos B + \sin B \cos A \qquad \text{let } A = B$$
$$\sin(A+A) = \sin A \cos A + \sin A \cos A$$
$$\boxed{\sin(2A) = 2\sin A \cos A}$$

Example 2: Find sin2θ when sinθ = 2/3, π/2 ≤ θ ≤ π.

$\sin^2\theta + \cos^2\theta = 1$

$(+\frac{2}{3})^2 + \cos^2\theta = 1$

$\cos\theta = \pm\sqrt{1 - \frac{4}{9}}$

$\cos\theta = \pm\sqrt{5/9}$

$\cos\theta = \mp\frac{\sqrt{5}}{3}$

Neg, 2nd Quad

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin 2\theta = 2(+\frac{2}{3})(-\frac{\sqrt{5}}{3})$

$\sin 2\theta = \boxed{\frac{-4\sqrt{5}}{9}}$

The identity for the sine of a half-angle is easily derived from cos2A = 1 – 2sin^{2}A:

$1 - 2\sin^2 A = \cos 2A$ let $2A = \theta$, $A = \frac{\theta}{2}$

$1 - 2\sin^2\frac{\theta}{2} = \cos\theta$

$2\sin^2\frac{\theta}{2} = 1 - \cos\theta$

$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$

Example 3: Find sin(A/2) when sinA = 4/5, π/2 ≤ A ≤ π.

$\sin^2 A + \cos^2 A = 1$

$(\frac{4}{5})^2 + \cos^2 A = 1$

$\cos A = \pm\sqrt{1 - \frac{16}{25}}$

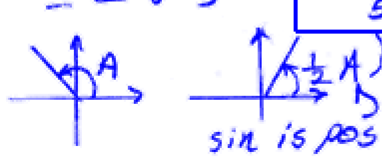
$\cos A = -\frac{3}{5}$

neg, 2nd Quad

$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$

$= \pm\sqrt{\frac{1-(-3/5)}{2}}$

$= \pm\sqrt{\frac{4}{5}} = \boxed{+\frac{2\sqrt{5}}{5}}$



Assignment:

1. Find $\sin(A + B)$ where $\sin A = -2/5$, $\pi \le A \le 3\pi/2$; $\cos B = 1/5$, $3\pi/2 \le B \le 2\pi$.

---

2. Find $\sin 2\theta$ when $\sin\theta = -1/2$, $3\pi/2 \le \theta \le 2\pi$.

---

3. Find $\sin(A/2)$ when $\cos A = -4/5$, $\pi/2 \le A \le \pi$.

4. Use the exact function values of 30^o, 45^o, and/or 60^o to find $\sin(105^o)$.

5. Use the exact function values of 30^o, 45^o, and/or 60^o to find $\sin(22^o\ 30')$.

6. Use the exact function values of 30^o, 45^o, and/or 60^o to find $\sin(120^o)$.

7. Consider evaluating sin(A + B). If the sines and cosines of *A* & *B* are known, what could be used to evaluate this composite function?

Prove the following identities:

8. $\sin 2A \cos A + \sin A \cos 2A = \sin 3A$

9. $2\sin\theta \cos\theta = \sin 4\theta \cos 2\theta - \sin 2\theta \cos 4\theta$

10. $\frac{\sin^2 2\theta}{1+\cos 2\theta} = 2\sin^2\theta$

11. $\frac{\sin 2\theta}{\sin\theta} - \frac{\cos 2\theta}{\cos\theta} = \sec\theta$

12. $\cot\theta + \tan\theta = 2\csc 2\theta$

Unit 4: Lesson 05 Tangent composite angles

To obtain the tangent of the sum of angles A and B, begin with

$$\tan(A + B) = \frac{\sin(A+B)}{\cos(A+B)}$$

Apply the expansion formulas for sin(A + B) and cos(A + B), simplify, and we are quickly led to (See **Enrichment Topic M**):

$$\tan(A + B) = \frac{\tan A+\tan B}{1-\tan A \tan B}$$

Similarly, it can be shown that

$$\tan(A - B) = \frac{\tan A-\tan B}{1+\tan A \tan B}$$

The two are often compactly written together as:

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Example 1: Find tan(A – B) if tan A = 3; cos B = –1/2, π/2 ≤ B ≤ π .

$$\tan(A-B) = \frac{3-(-\sqrt{3})}{1+3(-\sqrt{3})}$$

$$= \frac{3+\sqrt{3}}{1-3\sqrt{3}} \cdot \frac{1+3\sqrt{3}}{1+3\sqrt{3}}$$

$$= \frac{3+9\sqrt{3}+\sqrt{3}+9}{1-27}$$

$$= \frac{12+10\sqrt{3}}{-26}$$

$$= \boxed{\frac{-6}{13} - \frac{5\sqrt{3}}{13}}$$

$$\sin^2 B + \cos^2 B = 1$$

$$\sin^2 B + \left(-\tfrac{1}{2}\right)^2 = 1$$

$$\sin B = \pm\sqrt{1-\tfrac{1}{4}}$$

$$= \pm\sqrt{3/4}$$

$$= \frac{\sqrt{3}}{2}$$

pos because of 2nd Quad

$$\tan B = \frac{\sin B}{\cos B} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$$

By substituting B = A into tan(A + B) = $\frac{\tan A+\tan B}{1-\tan A\tan B}$, the formula for tan(2A) is quickly produced (See **Enrichment Topic M**):

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

Example 2: Find tan(2ϕ) when sinϕ = .2, 0 ≤ ϕ ≤ π/2. Give the answer in decimal form.

$\sin^2\phi + \cos^2\phi = 1$

$(.2)^2 + \cos^2\phi = 1$

$\cos\phi = \pm\sqrt{1-.04}$

$\cos\phi = .979795$ pos, 1st Quad

$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{.2}{.979795}$

$\tan\phi = .20412$

$\tan 2\phi = \frac{2(.20412)}{1-(.20412)^2}$

$= \boxed{.425998}$

Without proof here (See **Enrichment Topic M**) we present the formula for the tangent of a half-angle:

$$\tan(\theta/2) = \frac{\sin\theta}{1+\cos\theta}$$

Example 3: Find tan(A/2) when cosA = $-\sqrt{3}/2$, π/2 ≤ A ≤ π

$\sin^2 A + \cos^2 A = 1$

$\sin^2 A + (-\sqrt{3}/2)^2 = 1$

$\sin A = \pm\sqrt{1-3/4}$

$= \frac{1}{2}$ pos, in 2nd Quad.

$\tan\left(\frac{A}{2}\right) = \frac{\frac{1}{2}}{1-\frac{\sqrt{3}}{2}} \cdot \frac{1+\frac{\sqrt{3}}{2}}{1+\frac{\sqrt{3}}{2}}$

$= \frac{\frac{1}{2}+\frac{\sqrt{3}}{4}}{1-3/4}$

$= \frac{\frac{1}{2}+\frac{\sqrt{3}}{4}}{1/4} = \boxed{2+\sqrt{3}}$

Assignment:

1. Find tan(A + B) when sinA = 3/4, $0 \le A \le \pi/2$; sin B = 1/2, $\pi/2 \le B \le \pi$.

---

2. Find tan(2A) when tanA = –2 .

---

3. Find tan(B/2) when sinB = –2/5, $3\pi/2 \le B \le 2\pi$.

4. Use the exact function values of 30^o, 45^o, and/or 60^o to find $\tan(105^o)$.

5. Use the exact function values of 30^o, 45^o, and/or 60^o to find $\tan(22^o\ 30')$.

6. Use the exact function values of 30^o, 45^o, and/or 60^o to find $\tan(120^o)$.

7. Consider evaluating $\tan(A + B)$. If A and B are unknown but the tangents of these angles are known, what could be used to evaluate this composite function?

Prove the following identities:

8. $\tan(2\pi - \theta) = -\tan\theta$

9. $\tan\theta \cot(.5\theta) = 1 + \sec\theta$

10. $\cot 4\theta + \tan 2\theta = \csc 4\theta$

11. $\tan(2\pi + A) = \tan A$

12. $1 + \sin 2\theta = \frac{(1+\tan\theta)^2}{1+\tan^2\theta}$

Unit 4: Lesson 06 Product and factor identities, reference angles

Without proof we offer the **product identities** beyond those we have already had. These new identities need not be memorized (that's what books are for… look them up); however, students should be familiar with them to the extent that they can be recognized when encountered in a problem.

$2\sin A \cos B = \sin(A + B) + \sin(A - B)$
$2\cos A \sin B = \sin(A + B) - \sin(A - B)$
$2\cos A \cos B = \cos(A + B) + \cos(A - B)$
$2\sin A \sin B = -\cos(A + B) + \cos(A - B)$

The following group is known as the **factor identities**:

$\sin C + \sin D = 2\sin((C + D)/2)\ \cos((C - D)/2)$
$\sin C - \sin D = 2\cos((C + D)/2)\ \sin((C - D)/2)$
$\cos C + \cos D = 2\cos((C + D)/2)\ \cos((C - D)/2)$
$\cos C - \cos D = -2\sin((C + D)/2)\ \sin((C - D)/2)$

Example 1: Prove $\dfrac{\sin 4x - \sin 2x}{\cos 4x + \cos 2x} = \tan x$

$$\frac{\cancel{2}\cos((4x+2x)/2)\ \sin((4x-2x)/2)}{\cancel{2}\cos((4x+2x)/2)\ \cos((4x-2x)/2)} = \tan x$$

$$\frac{\cancel{\cos 3x}\ \sin x}{\cancel{\cos 3x}\ \cos x} = \tan x$$

$$\tan x \overset{\checkmark}{=} \tan x$$

Example 2: Prove cos2x – cos4x = 2sinx sin3x

$$\cos 2x - \cos 4x = -\cos(x+3x) + \cos(x-3x)$$

$$\cos 2x - \cos 4x = \cos(-2x) - \cos(4x) \qquad \cos(-\theta) = \cos\theta$$

$$\cos 2x - \cos 4x \overset{\checkmark}{=} \cos(2x) - \cos(4x)$$

Example 3: Express this product of trig functions as a sum of trig functions and simplify: 2cos6θ cos4θ

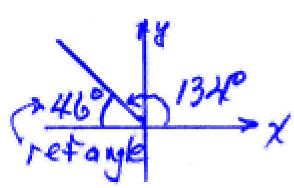
$$2\cos 6\theta \cos 4\theta = \cos(6\theta + 4\theta) + \cos(6\theta - 4\theta)$$
$$= \boxed{\cos(10\theta) + \cos(2\theta)}$$

Applying the identities for sin(A ± B), cos(A ± B), and tan(A ± B) it can be shown that **any trig function of an angle is equal to the adjusted sign of its reference angle (also called related angle)**.

The **reference angle** of an angle is the smallest angle the terminal side makes with the x-axis.

In the following examples, draw the angle and its reference angle. Then express the given function in terms of the same function and with its sign adjusted.

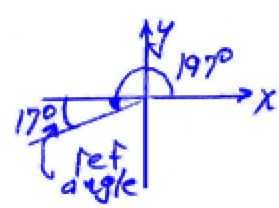
Example 4: sin(134°)



$$\boxed{\sin 134° = \sin 46}$$

pos because in 2nd Quad

Example 5: cos(197°)



$$\cos(197°) = -\cos(17°)$$

Neg because in 3rd Quad.

Assignment: In the following problems, draw the angle and its reference angle. Then express the given function in terms of the same function and with its sign adjusted. Assume angles that are not specifically designated in degrees are in radians.

1. $\tan(119^\circ)$	2. $\csc(7\pi/6)$
3. $\cos(342^\circ)$	4. $\sin(189^\circ)$
5. $\cot(22^\circ)$	6. $\sec(147^\circ)$

Express each product of trig functions as a sum or difference of trig functions and simplify.

7. $3\cos 52^\circ \sin 14^\circ$

8. sinx sin2x

9. $2\sin 45^\circ \cos 18^\circ$

10. $\cos 7\pi \cos(-2\pi)$

Prove the following trig identities.

11. $\dfrac{\cos 3A - \cos 5A}{\sin 3A + \sin 5A} = \tan A$

12. $2\sin 3x \sin x = (1 - \cos 2x)(1 + 2\cos 2x)$

Unit 4: Cumulative Review

1. Combine into a single fraction and simplify $\frac{2x}{x^2-9}+\frac{-4}{x+3}$

2. Simplify $\dfrac{\frac{x+3}{4}+2}{\frac{x}{12}-\frac{3}{2}}$

3. Find the area of triangle ABC in which a = 4.5, b = 7, C = 34°.

4. Write the sine law in terms of A, B, C, a, b, and c.

5. Write the cosine law in terms of A, B, C, a, b, and c.

6. Define sin, cos, and tan in terms of x, y, and r.

7. What is the equation of a line in point slope form that passes through (–5, 1) and is parallel to the line given by $x/3 + y/7 = 1$?

8. Use a graphing calculator to evaluate $4\cos(\pi/7) + \sin(\ 2(\pi - 2.5)\ )$ where all angles are in radians.

9. Convert 36^{o} to radians.

10. Two force vectors, when placed tail-to-tail, have an angle of 40^{o} between them. One vector has a magnitude of 23 newtons while the other's magnitude is 52 newtons. What is the magnitude of the resultant and what angle does it make with the larger force?

11. Draw a 30-60-90 triangle and label the standard lengths of the sides.

12. Draw a 45-45-90 triangle and label the standard lengths of the sides.

13. Solve the triangle ABC where $a = 13$, $C = 33^{o}$, $B = 101^{o}$

14. Solve the triangle ABC where $a = 12$, $b = 7.2$, $B = 21^{o}$

Unit 4: Review

1. Complete these identities:

   $\sin^2 A + \cos^2 A = ?$

   $1 - \sec^2 A = ?$

   $1 + \cot^2 A = ?$

2. Complete these identities:

   $\sin B \csc B = ?$

   $1/\sec B = ?$

   $1/\tan B = ?$

3. Prove $(1 - \cos\theta)(1 + \cos\theta) = \sin^2\theta$

4. Prove $\cot A/\cos A = \csc A$

5. Complete these identities:

   $\cos(A - B) = ?$

   $\sin(A + B) = ?$

Using the exact function values for 30°, 60°, and 45°, find the indicated trig functions in problems 6 & 7:

6. $\tan(15^\circ)$

7. $\cos(105^\circ)$

8. Complete these identities:

$\tan(A - B) = ?$

$\cot(2A) = ?$

$\tan(A/2) = ?$

9. If $\pi \le B \le 3\pi/2$, in which quadrant is the terminal side of angle B?

10. Find $\cos(A - B)$ when $\sin A = 3/5$, $0 \le A < \pi/2$; $\cos B = -1/5$, $\pi/2 \le B \le \pi$

11. Find $\tan(A + B)$ when $\tan A = 2$, $\pi \le A < 3\pi/2$; $\sin B = 1/5$, $\pi/2 \le B \le \pi$

12. Show that $\sin(\theta + \pi) = -\sin\theta$

---

13. Prove $\frac{\sin 2\theta}{\sin\theta} - \frac{\cos 2\theta}{\cos\theta} = \sec\theta$

---

14. Prove $\tan 2\theta - \tan\theta = \tan\theta \sec 2\theta$

15. Complete this identity:

$\sin(-\theta) = ?$

16. Complete this identity:

$\cos(-\theta) = ?$

In the following two problems, draw the angle and its reference angle. Then express the given function in terms of the same function of the reference angle and with its sign adjusted.

17. $\sin(197^{o})$

18. $\cos(328^{o})$

19. Prove $\dfrac{\cos 3x - \cos 5x}{\sin 3x + \sin 5x} = \tan x$

Identities that must be memorized:

$\sin^2 A + \cos^2 A = 1$

$1 + \tan^2 A = \sec^2 A$

$1 + \cot^2 A = \csc^2 A$

$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$

$\cos(A \pm B) = \cos A \cos B \mp \sin B \sin A$

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$\sin 2A = 2 \sin A \cos A$

$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$

$\tan A = \frac{\sin A}{\cos A}$

$\sin A \csc A = 1$ $\quad$ $\cos A \sec A = 1$ $\quad$ $\tan A \cot A = 1$

$\sin(-A) = -\sin A$ $\quad$ $\cos(-A) = \cos A$ $\quad$ $\tan(-A) = -\tan A$

Identity sheet available on tests:

$\cos 2A = \cos^2 A - \sin^2 A$

$\cos 2A = 1 - 2\sin^2 A$

$\cos 2A = 2\cos^2 A - 1$

$\sin(A/2) = \pm\sqrt{\frac{1-\cos A}{2}}$

$\cos(A/2) = \pm\sqrt{\frac{1+\cos A}{2}}$

$\tan(A/2) = \frac{\sin A}{1+\cos A}$

$2\sin A \cos B = \sin(A + B) + \sin(A - B)$

$2\cos A \sin B = \sin(A + B) - \sin(A - B)$

$2\cos A \cos B = \cos(A + B) + \cos(A - B)$

$2\sin A \sin B = -\cos(A + B) + \cos(A - B)$

$\sin C + \sin D = 2\sin((C + D)/2) \cos((C - D)/2)$

$\sin C - \sin D = 2\cos((C + D)/2) \sin((C - D)/2)$

$\cos C + \cos D = 2\cos((C + D)/2) \cos((C - D)/2)$

$\cos C - \cos D = -2\sin((C + D)/2) \sin((C - D)/2)$

Pre Calculus, Unit 5

Solving Trigonometric Equations

Unit 5: Lesson 01 Simple trig equations

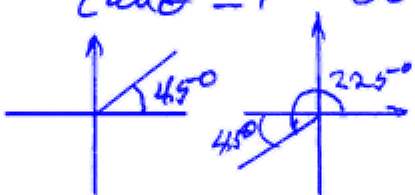
A trigonometric equation is an equation that involves at least one trig function.

In this lesson we will consider equations with only one trig function and one angle: (Also, all solutions will be limited to $0 \le \theta < 360^o$.)

Example 1: Solve $5\tan\theta - 2 - 4\tan\theta = -1$

$5\tan\theta - 4\tan\theta = 2 - 1$

$\tan\theta = 1 \qquad \theta = \tan^{-1}(1) = 45^\circ$ ← This is the reference angle



$\theta = 45^\circ, 225^\circ$

Two places where tan is pos.

Example 2: Solve $2\sin^2\theta - 3\sin\theta - 2 = 0$

Quadratic in $\sin\theta$

$a = 2 \quad b = -3 \quad c = -2$

$$\sin\theta = \frac{3 \pm \sqrt{9 - 4(2)(-2)}}{2 \cdot 2} = \frac{3 \pm \sqrt{25}}{4} = \frac{3 \pm 5}{4} = 2, -\frac{1}{2}$$

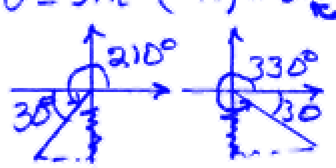
$\sin\theta = -\frac{1}{2}$

$\theta = \sin^{-1}(\frac{1}{2}) = 30^\circ$ reference angle

$\sin\theta = 2$

reject

sine must be between -1 and 1



Two places where sine is neg.

$\theta = \boxed{210^\circ, 330^\circ}$

Now consider equations with one member factorable and the other zero:

Example 3: tan2θ sinθ – sinθ – tan2θ + 1 = 0

$\tan(2\theta)\sin\theta - \tan(2\theta) - \sin\theta + 1 = 0$

Factor by grouping

$\{\tan(2\theta)[\sin\theta - 1] - 1[\sin\theta - 1]\} = 0$

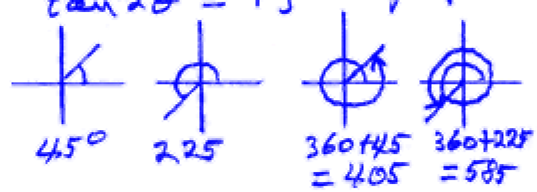
$(\sin\theta - 1)[\tan(2\theta) - 1] = 0$

$\sin\theta - 1 = 0$

$\sin\theta = 1$

$\theta = \boxed{90^\circ}$

$\left.\begin{array}{l}\tan 2\theta - 1 = 0 \\ \tan 2\theta = 1\end{array}\right\}$ $2\theta = 45^\circ$ ref angle



$2\theta = 45$

$\theta = \boxed{22.5}$

$2\theta = 225$

$\theta = \boxed{112.5}$

$2\theta = 405$

$\theta = \boxed{202.5}$

$2\theta = 585$

$\theta = \boxed{292.5}$

Assignment: Solve the following equations while limiting the solutions to $0 \le \theta < 360^\circ$.

1. $3\sec^2\theta - 4 = 0$

2. $\tan\theta - \sqrt{3} = 0$

3. $2\sin\theta - 1 = 0$

4. $3\sin 2\theta = 1 + 2\sin 2\theta$

5. $\tan^2\theta + \tan\theta = 0$

6. $2\sqrt{3}\cos^2\theta + \cos\theta = 2\sqrt{3}$

7. $\cos^2\theta + 2\cos\theta - 3 = 0$

8. $2\sin\theta\cos\theta + \sin\theta = 0$

9. $\cos\theta\tan\theta - \sqrt{3}\cos\theta + \tan\theta - \sqrt{3} = 0$

Unit 5: Lesson 02 **Advanced trig equations**

Equations made solvable or factorable by a substitution:

Example 1: Solve $4\tan^2A - 3\sec^2A = 0$

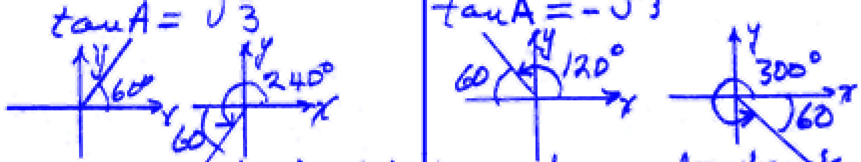
$4\tan^2A - 3(1+\tan^2A) = 0$ sub

$4\tan^2A - 3 - 3\tan^2A = 0$

$\tan^2A = 3$

$\tan A = \pm\sqrt{3}$ → ref angle $= 60°$

$\tan A = \sqrt{3}$ | $\tan A = -\sqrt{3}$



Two places where tan is pos | Two places where tan is neg.

$\theta = \boxed{60°, 240°}$ | $\theta = \boxed{120°, 300°}$

Equations of the form Asinθ + Bcosθ = C:

Example 2: Solve sinθ + 3 cosθ = .8

$(3\cos\theta)^2 = (.8 - \sin\theta)^2$ must check because of squaring.

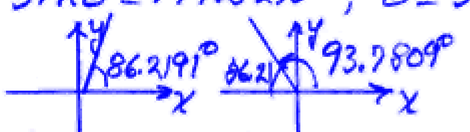
$9\cos^2\theta = .64 - 1.6\sin\theta + \sin^2\theta$

$9(1-\sin^2\theta) = .64 - 1.6\sin\theta + \sin^2\theta$

$-10\sin^2\theta + 1.6\sin\theta + 8.36 = 0$

$\sin\theta = \dfrac{-1.6 \pm \sqrt{1.6^2 - 4(-10)8.36}}{2(-10)} = -.83782, .9978235$

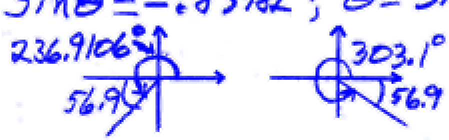
$\sin\theta = .9978235$; $\theta = \sin^{-1}(.9978235) = 86.2191°$ ref angle



Two places where sin is pos

$\theta = \boxed{\cancel{86.2191°}, 93.7809°}$ reject

$\sin\theta = -.83782$; $\theta = \sin^{-1}(-.83782) = -56.9106°$



$\theta = \boxed{\cancel{236.9106°}\ 303.0893°}$ reject

Assignment: Solve these equations for the given variable. Limit all answers to the range $0^o \le \theta < 360^o$.

1. $\sin^2\theta - \cos^2\theta = 0$

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2. $2\cos\theta - 5 + 2\sec\theta = 0$

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3. $\cos 2\theta - \sin\theta = 0$

4. $\sin\theta - \sqrt{3}\cos\theta = 1$

5. $\csc\theta - 3 + 2\sin\theta = 0$

6. $\cos 2\theta = 2\sin^2\theta$

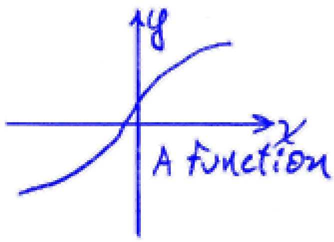
7. $\sin\theta + \cos\theta = .2$

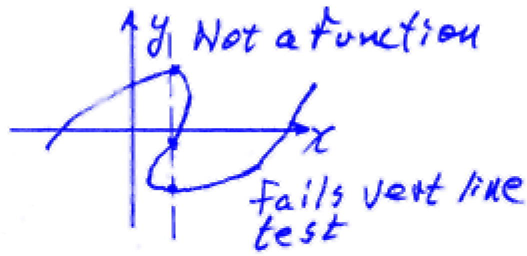
Pre Calculus, Unit 6

Function Fundamentals

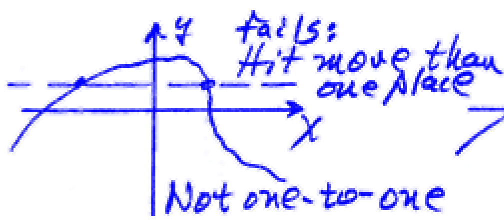
Unit 6: Lesson 01 Basic definitions

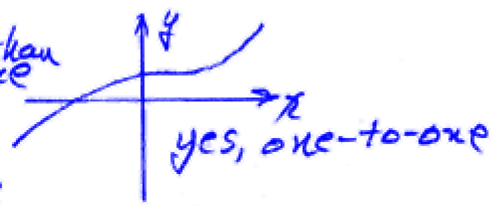
Function definition: A relation (set of points) in which each x-value is assigned to only one y-value… it passes the vertical line test.





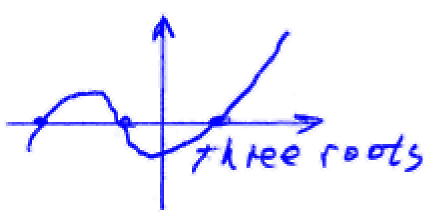
One-to-one relation: A relation (set of points) in which each y-value is assigned to only one x-value. It passes the horizontal line test.



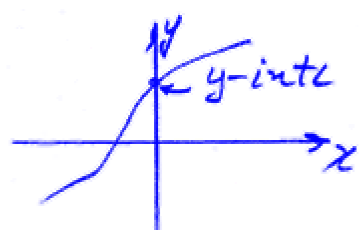


X-intercept: Point on the graph where the relation touches the x-axis (y = 0).

Goes by two other names: **root** and **zero**



Y-intercept: Point on the graph where the relation touches the y-axis (x = 0).



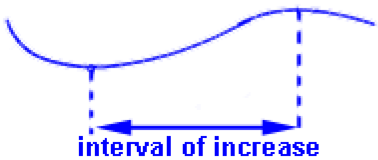
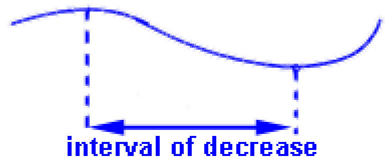
Interval notation: Intervals can be shown using inequalities or with parenthesis (**not equal**) and square brackets (**equal**):

Example 1: $-\infty < x \leq 5$

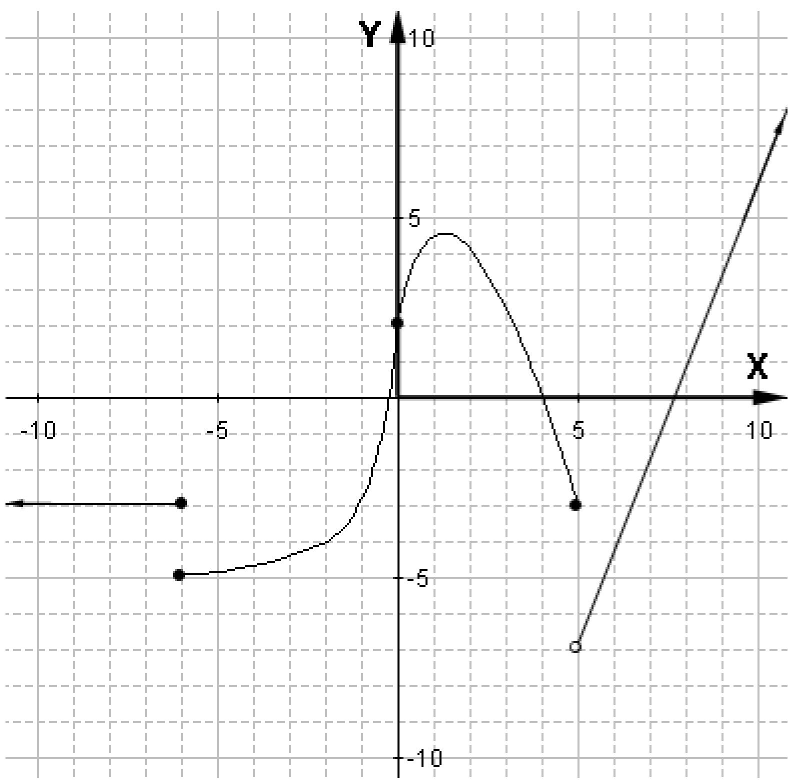
$(-\infty, 5]$

Example 2: $7 \leq x < 19$

$[7, 19)$

Increasing : A relation increases on an interval if as *x* increases, *y* increases. 	**Decreasing :** A relation decreases on an interval if as *x* increases, *y* decreases. 
Constant: A relation is constant over an interval if it remains at the same value.	

Example 3:



a. Is this a function?
No

b. Y-intercept?
2

c. Zeros?
-2, 4, 7.7

d. f(5)?
-3

e. One-to-one?
No

f. f(0)?
2

g. Increasing interval(s)?
(-6,1.2), (5,∞)

h. Decreasing interval(s)?
(1.2,5)

i. Constant interval(s)?
(-∞,-6)

j. Intervals where f(x) ≥ 0
[-2, 4], [7.7,∞)

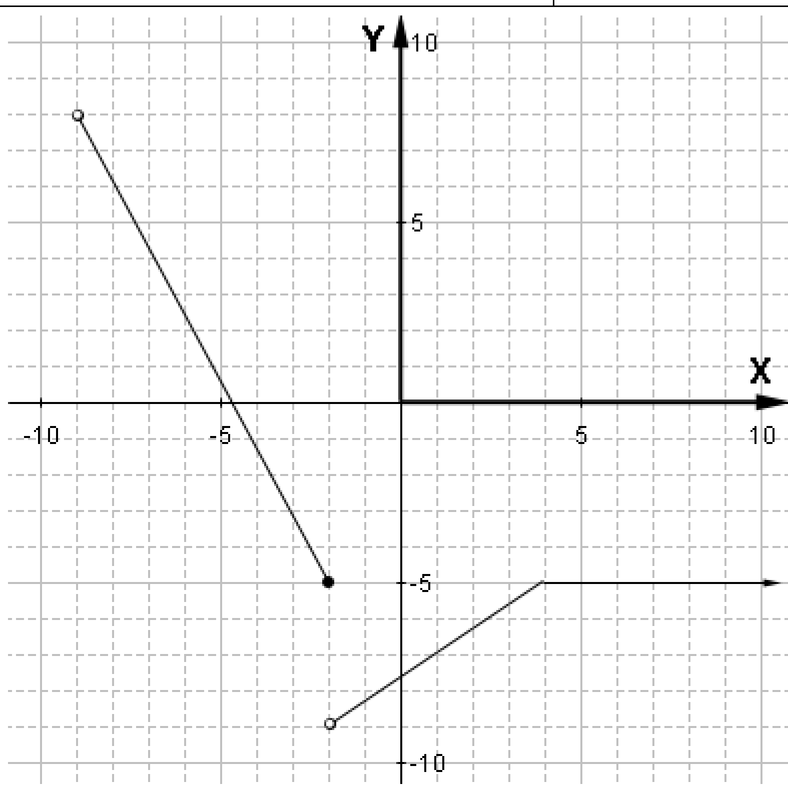
Domain: x values used All real x

Range: y values used (-7,∞)

Assignment:

1. Express the interval $-\infty < x < 57.2$ in "([])" notation.	2. Express the interval $7 \le x \le 59$ in "([])" notation.
3. Express the interval (4, 7] in "inequality" notation.	4. Express the interval [–11, 22.2) in "inequality" notation.

5.



a. Is this a function?

b. Y-intercept?

c. X-intercepts?

d. f(6)?

e. One-to-one?

f. f(–2)?

g. Incr interval(s)?

h. Dec interval(s)?

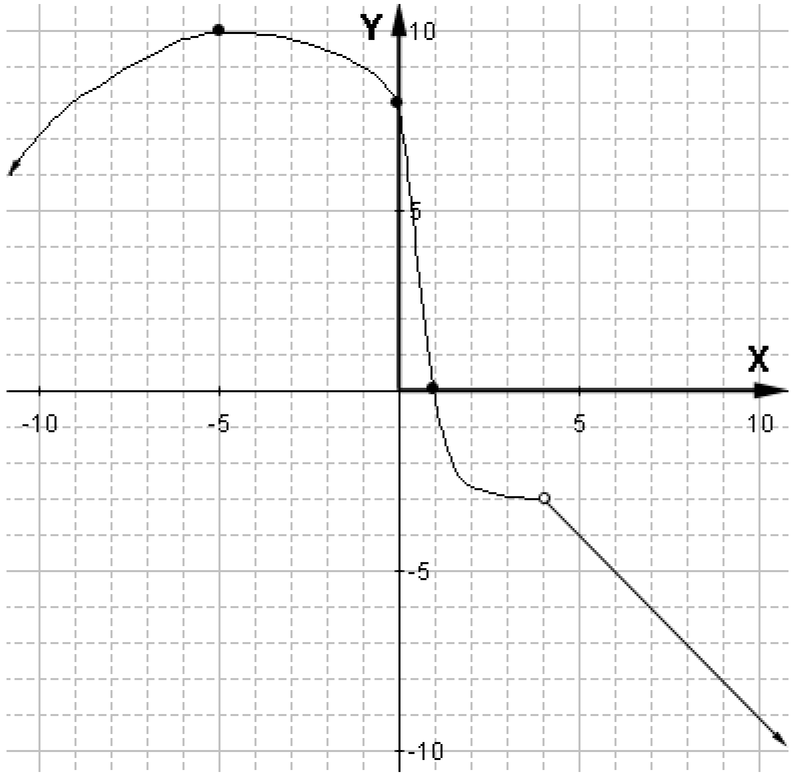
i. Constant interval(s)?

j. Interval where f(x) <0

k. Domain?

l. Range?

6.



a. Is this a function?

b. Y-intercept?

c. Roots?

d. f(1)?

e. One-to-one?

f. f(–9)?

g. Incr interval(s)?

h. Dec interval(s)?

i. Constant interval(s)?

j. Interval where f(x)< 0

k. Domain?

l. Range?

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7. At what x and y values is the function above undefined? Why?

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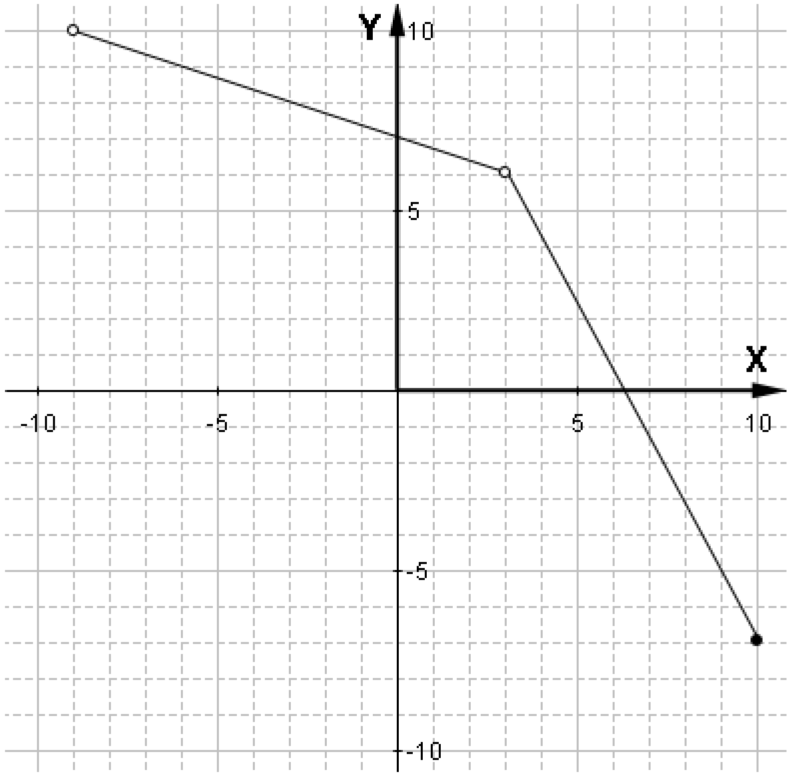
8. Draw an example of a relation that is neither a function nor one-to-one.	9. Draw an example of a relation that is both a function and one-to-one.

10.



a. Is this a function?

b. Y-intercept?

c. Roots?

d. f(9)?

e. One-to-one?

f. f(–3)?

g. Incr interval(s)?

h. Dec interval(s)?

i. Constant interval(s)?

j. Interval where f(x) >0

k. Domain?

l. Range?

11. Draw a function that that has 3 zeros and a y-intercept at y = 3.	12. Draw a relation that has two y-intercepts and one x-intercept. Is it a function?
13. Express the interval –6 ≤ x < ∞ in "([])" notation.	14. Express the interval [11, 101) in "inequality" notation.

Unit 6: Lesson 02 More on domain, intercepts, notation, function values

Domain:

The following are but a few of the x-values that must be excluded from the **natural** domain of a function.

- A value of *x* that causes a denominator to be zero.
- A value of *x* that causes the quantity under an even root to be negative.
- A value of *x* that causes the argument or base of a logarithm to be negative or zero.

In addition to the above functional requirements, practical requirements in word problems will often **artificially** restrict the domain. For example,

- the weight of an airplane can't be negative,
- the number of dogs must be a positive integer,
- the sine of an angle must be between –1 and 1, etc.

Intercepts:

- To determine x-intercepts, set y = 0 and solve for *x.*
- To determine y-intercepts, set x = 0 and solve for *y.*

In the following examples, determine the domain and the intercepts:

Example 1: y = 5x + 2

Domain: $\boxed{ARX}$
(All real x)

x-intc
$0 = 5x + 2$
$5x = -2$
$x = \boxed{-\frac{2}{5}}$

y-intc
$y = 5(0) + 2$
$y = \boxed{2}$

Example 2: y = 4x/(x – 3)

Domain: $\boxed{x \neq 3}$

x-intc
$0 = 4x/(x-3)$
$4x = 0$
$x = \boxed{0}$

y-intc
$y = \frac{4(0)}{(0-3)} = \frac{0}{-3}$
$y = \boxed{0}$

Example 3: $y = \sqrt{x+2}$

Domain: $x+2 \geq 0$
$\boxed{x \geq -2}$

x-intc
$(0)^2 = (\sqrt{x+2})^2$
$0 = x+2$
$x = \boxed{-2}$

y-intc
$y = \sqrt{0+2}$
$y = \boxed{\sqrt{2}}$

Example 4: $y = \log_b(x - 7)$

Domain: $x-7>0$
$\boxed{x>7}$

x-intc
$0 = \log_b(x-7)$
$x-7 = b^0$
$x-7 = 1$
$x = \boxed{8}$

y-intc
$y = \log_b(0-7)$
illegal!
No intercept!

Notation:
The notation f(–2) means to evaluate the function *f* at –2 (substitute in –2).

Example 5: If f(x) = x^2 –3x, what is f(–2)?

$f(-2) = (-2)^2 - 3(-2)$
$= 4 + 6$
$= \boxed{10}$

Example 6: If f(x) = (x +3)/x, name the ordered pair given by f(4).

$x = 4$
$y = f(4) = (4+3)/4$
$= 7/4$
$(x, y) = \boxed{(4, 7/4)}$

Example 7: Where does f(x) = x + 7 cross the x-axis?

x-intc $\rightarrow$ $y=0$

$y = x+7$

$0 = x+7$

$x = \boxed{-7}$

Example 8: If f(x) = (x – 3)/(x + 4) where does f(x) = 5?

$\frac{5}{1} = \frac{x-3}{x+4}$

$5(x+4) = x-3$

$5x+20 = x-3$

$5x-x = -3-20$

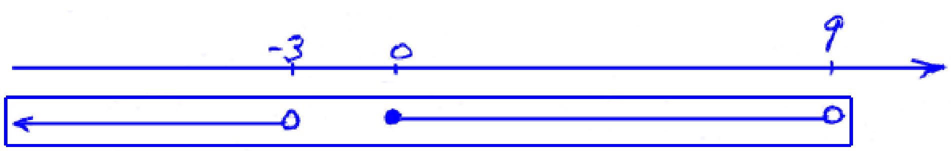
$4x = -23$

$x = \boxed{-\frac{23}{4}}$

Another way to specify discontinuous domain intervals (or any other intervals):

Suppose the domain of a function is given by [-4.2, 2), (5, ∞). This is the union of two intervals and is often given by[-4.2, 2) ∪ (5, ∞).

Example 9: On a number line, draw the points given by (–∞, -3) ∪ [0, 9).



See **Calculator Appendix F** for how to restrict the domain of a function on a graphing calculator.

Assignment: In problems 1-4, determine the domain, x-intercept, and y-intercept for each function.

1. $y = (x + 5)/x$

2. $y = 4\log_3(x + 17)$

3. $y = (x+ 2)/(x^2 - 16)$

4. $y = \sqrt[3]{x-7}/(x^2 + x - 6)$

In problems 5-12, use f(x) = (x – 9)/(x + 1).

5. Find f(5) .	6. What value of x is excluded from the domain of f(x)?
7. Where does f(x) cross the vertical axis?	8. What are the zero(s) of f?
9. Where does f = 18?	10. Evaluate 4f(–1).
11. Does the point (5, –2) lie on the graph of f(x)?	12. Does the point (1, –4) lie on the graph of f(x)?

13. On a number line, draw the intervals given by $(-\infty, -3] \cup -1 \cup (4, \infty)$.

Unit 6: Lesson 03 Function operations, composite functions

The following notation is used for operations on functions and the composition of functions:

- (f + g)(x) = f(x) + g(x) ; sum of functions
- (f – g)(x) = f(x) – g(x); difference of functions
- (fg)(x) = f(x) g(x); product of functions
- (f/g)(x) = f(x)/g(x); quotient of functions
- (f ○ g)(x) = f(g(x)); composite functions

In the following examples, assume that $f(x) = x^2 - 4x + 1$ and $g(x) = 3x - 2$ and perform the indicated operation.

Example 1: (f + g)(x)

$= f(x) + g(x)$
$= x^2 - 4x + 1 + 3x - 2$
$= \boxed{x^2 - x - 1}$

Example 2: (f – g)(x)

$= f(x) - g(x)$
$= x^2 - 4x + 1 - (3x - 2)$
$= x^2 - 4x + 1 - 3x + 2$
$= \boxed{x^2 - 7x + 3}$

Example 3: (f g)(x)

$= f(x)\,g(x)$
$= (x^2 - 4x + 1)(3x - 2)$
$= 3x^3 - 12x^2 + 3x - 2x^2 + 8x - 2$
$= \boxed{3x^3 - 14x^2 + 11x - 2}$

Example 4: (f/g)(x)

$= f(x)/g(x)$
$= \boxed{(x^2 - 4x + 1)/(3x - 2)}$

We could divide, but just as simple to leave it like this.

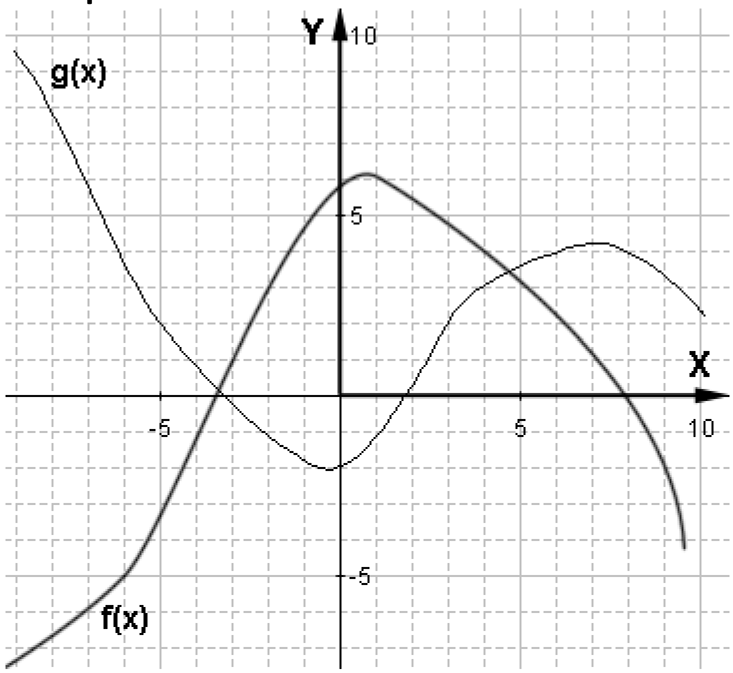
Example 5: $(f \circ g)(x)$

$= f(g(x)) = (3x-2)^2 - 4(3x-2) + 1$

$= 9x^2 - 12x + 4 - 12x + 8 + 1$

$= \boxed{9x^2 - 24x + 13}$

Example 6:



a. $(f + g)(-4)$

$= f(-4) + g(-4)$

$= -1 + 1 = \boxed{0}$

b. $(f - g)(-5)$

$= f(-5) - g(-5)$

$= -3 - 2 = \boxed{-5}$

c. $(f/g)(4)$

$= f(4)/g(4)$

$= \boxed{4/3}$

d. $g(f(-6))$

$= g(-5) = \boxed{2}$

e. $g(f(-2))$

$= g(3) = \boxed{2}$

f. $(f \circ g)(4)$

$= f(g(4)) = f(3)$

$= \boxed{4.8}$

Assignment: In problems 1-6, assume that $f(x) = 4/(x^2 – 9)$ and $g(x) = x/(x – 3)$ and perform the indicated operation.

1. $(f + g)(–2)$

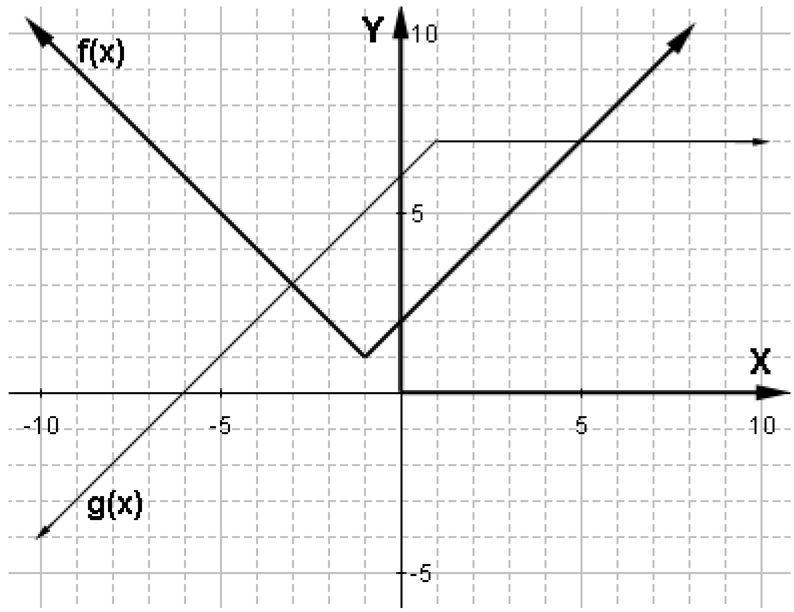
2. $(f – g)(–1)$

3. $(f/ g)(x)$

4. $g(f(x))$

5. $(f \circ g)(x)$

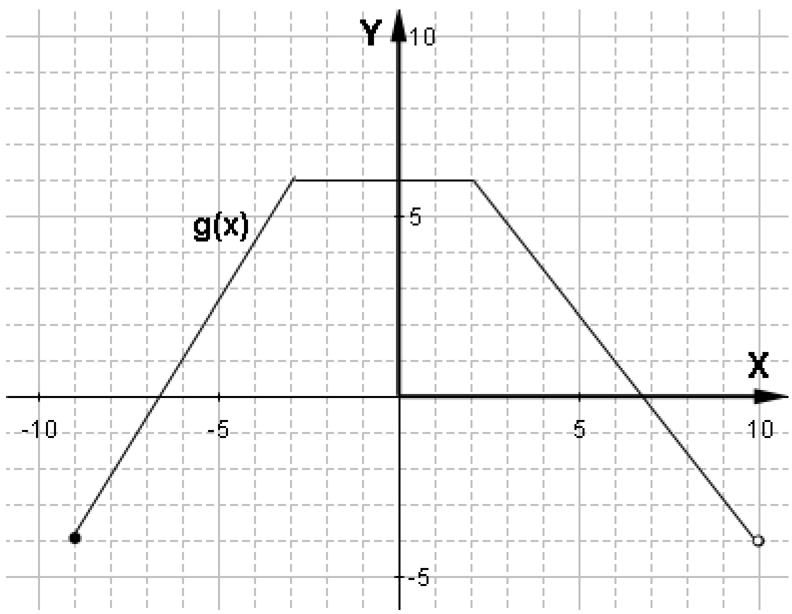
6. $f(2) + g(–1)$



Use f(x) and g(x) in problems 7-16.

7. $f(g(-4))$	8. $(g \circ f)(2)$
9. $(f + g)(6)$	10. $f(3) - g(-5)$
11. $3f(2) - 5g(-1)$	12. Where are f and g equal?

13. What is(are) the interval(s) over which $f(x) \geq g(x)$?	14. What is(are) the interval(s) over which $f(x) - g(x) > 0$?
15. Express the answer to #13 using ∪ notation.	16. Express the answer to #14 using ∪ notation.

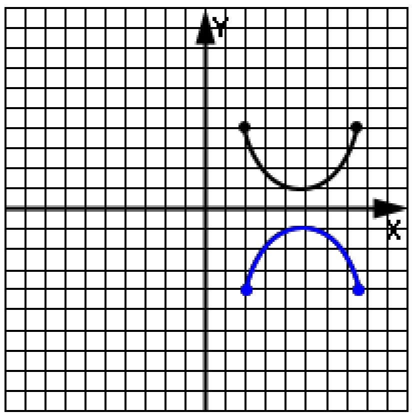


Use this g(x) in problems 17-20.

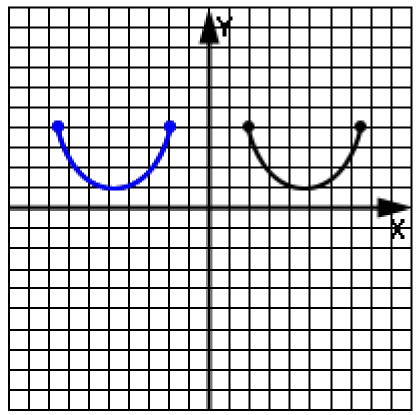
17. Where does $g(x) = 1$?	18. Over what interval is $g(x)$ increasing?
19. Over what interval is $g(x)$ constant?	20. What is the domain of $g(x)$?

Unit 6: Lesson 04 Reflections

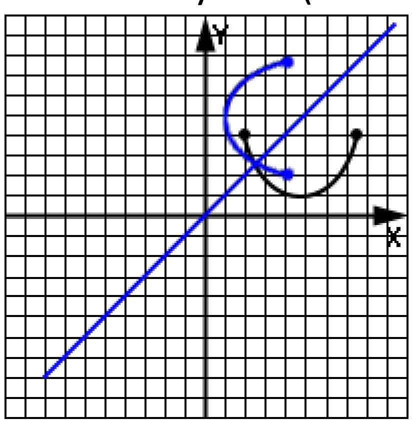
Example 1:
Graphically reflecting a function across the x-axis:



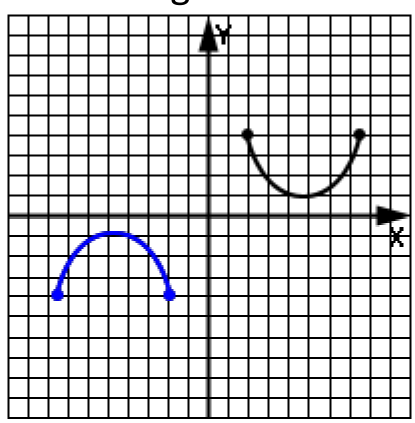
Example 2:
Graphically reflecting a function across the y-axis:



Example 3:
Graphically reflecting a function across the line y = x (45° line):



Example 4:
Graphically reflecting a function across the origin:



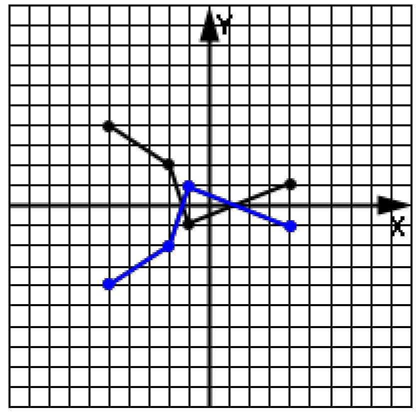
Reflections of the type done in example 3 above are important in the study of **inverse functions**.

Notice that the first three examples above produce **reflections** of the original function **across a line**.

Example 5: Using a table of points, produce a **reflection across the x-axis.** Make a new table in which we **change the signs of the y-values**.

x	y
–5	4
–2	2
–1	–1
4	1

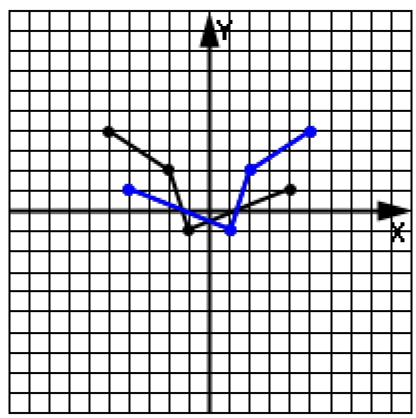
x	y
–5	**–4**
–2	**–2**
–1	**1**
4	**–1**



Example 6: Using a table of points, produce a **reflection across the y-axis.** Make a new table in which we **change the signs of the x-values**.

x	y
–5	4
–2	2
–1	–1
4	1

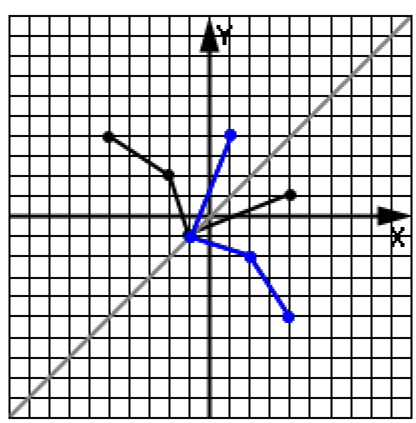
x	y
5	**4**
2	**2**
1	**–1**
-4	**1**



Example 7: Using a table of points, produce a **reflection across the line y = x**. Make a new table in which we **interchange the x and y-values**.

x	y
–5	4
–2	2
–1	–1
4	1

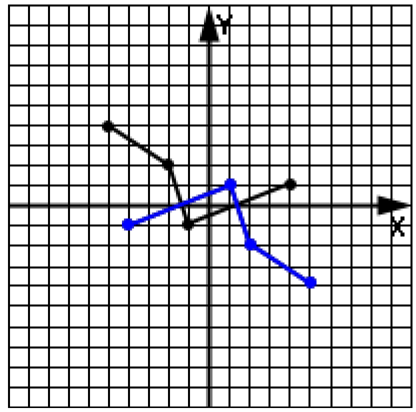
x	y
4	**–5**
2	**–2**
-1	**–1**
1	**4**



Example 8: Using a table of points, produce a **reflection across the origin**. Make a new table in which we **change the signs of both x and y**.

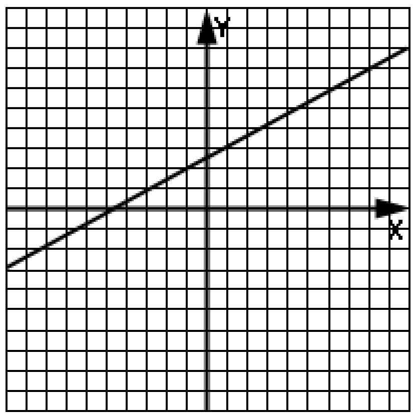
x	y
–5	4
–2	2
–1	–1
4	1

x	y
5	**–4**
2	**–2**
1	**1**
–4	**–1**

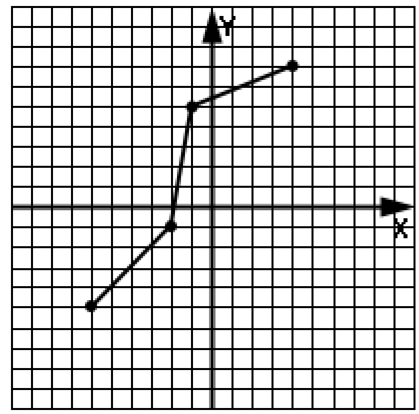


Assignment:

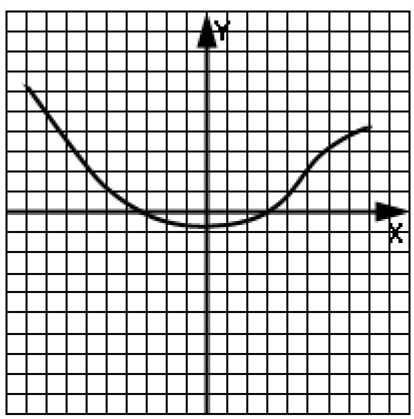
1. Sketch the reflection of the function across the x-axis:



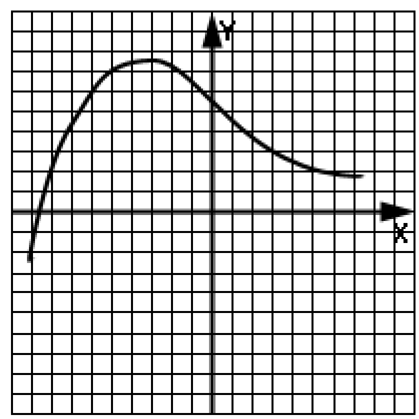
2. Sketch the reflection of the function across the x-axis:



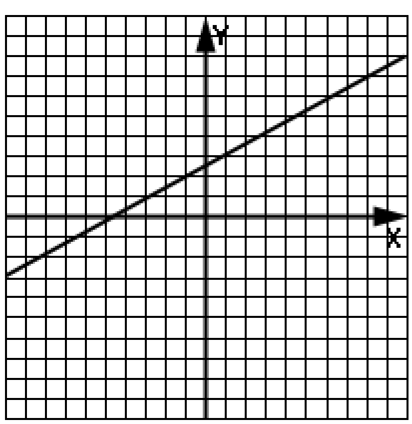
3. Sketch the reflection of the function across the x-axis:



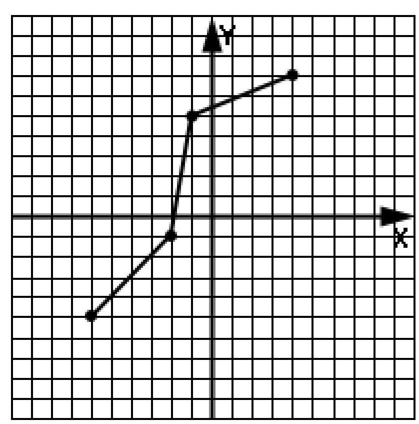
4. Sketch the reflection of the function across the x-axis:



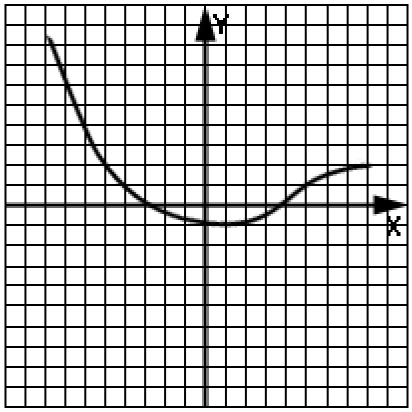
5. Sketch the reflection of the function across the y-axis:



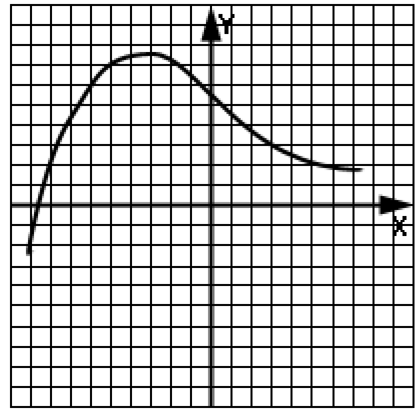
6. Sketch the reflection of the function across the y-axis:



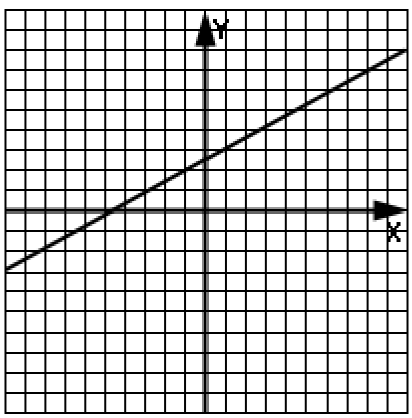
7. Sketch the reflection of the function across the y-axis:



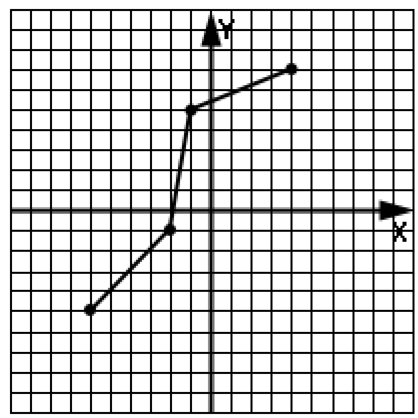
8. Sketch the reflection of the function across the y-axis:



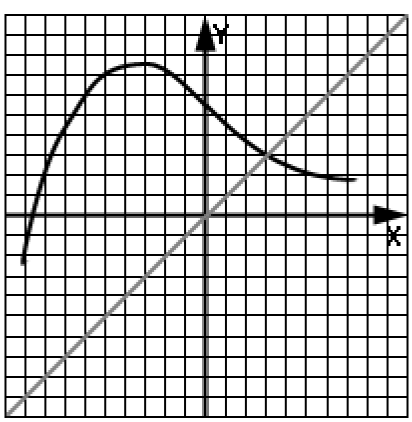
9. Sketch the reflection of the function across the line given by y = x:



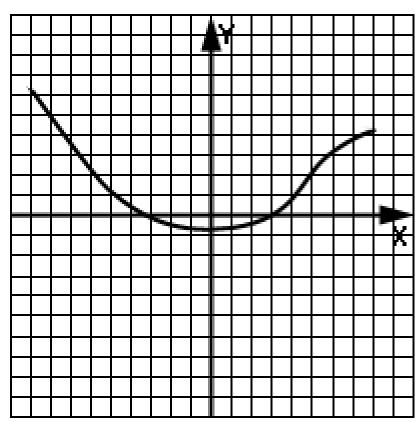
10. Sketch the reflection of the function across the origin:



11. Sketch the reflection of the function across the line given by y = x:



12. Sketch the reflection of the function across the origin:

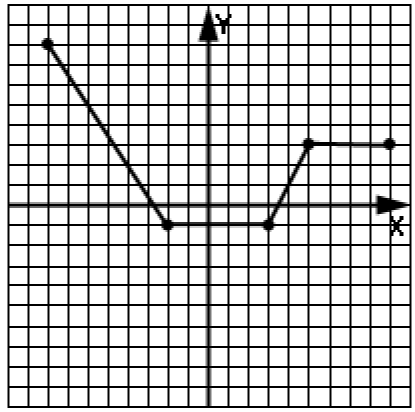


Fill in the blanks of the second table so as to produce the indicated reflection. Sketch the reflection alongside the graph of the provided graph of the original function.

13. Reflection across the x-axis

x	y
–8	8
–2	–1
3	–1
5	3
9	3

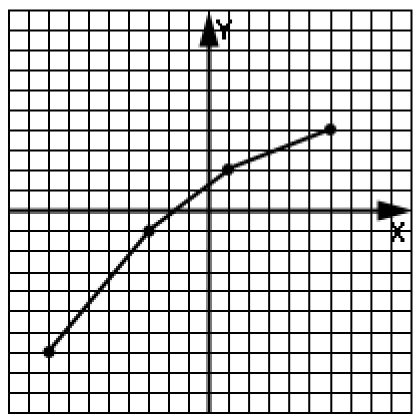
x	y



14. Reflection across the y-axis

x	y
–8	–7
–3	–1
1	2
6	4

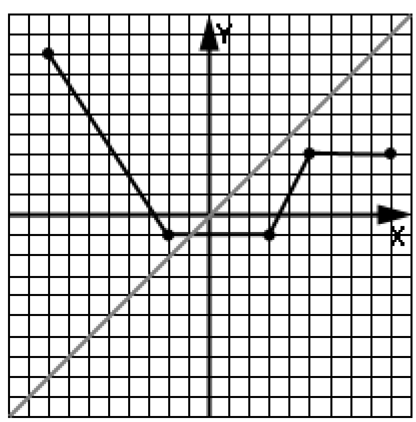
x	y



15. Reflection across the line y = x

x	y
–8	8
–2	–1
3	–1
5	3
9	3

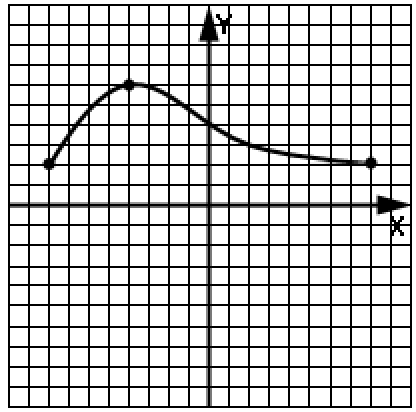
x	y



16. Reflection across the x-axis

x	y
–8	2
–4	6
8	2

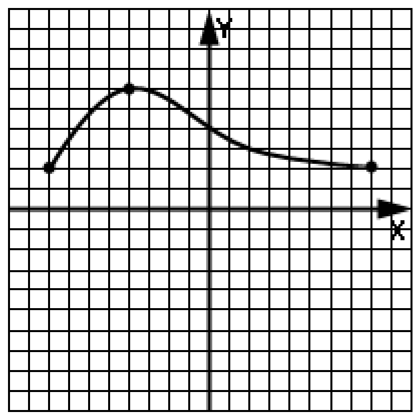
x	y



17. Reflection across the y-axis

x	y
–8	2
–4	6
8	2

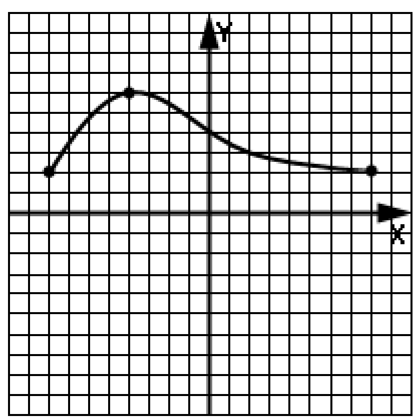
x	y



18. Reflection across the origin

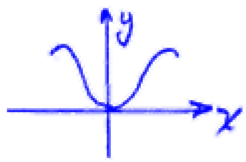
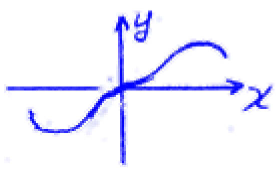
x	y
–8	2
–4	6
8	2

x	y



Unit 6: Lesson 05 Even and odd functions

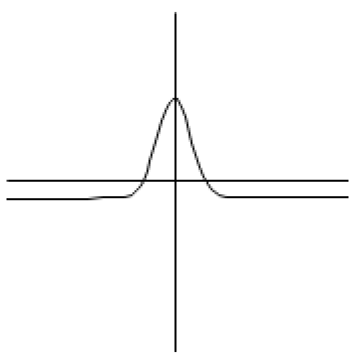
Functions can fall into the category of **even**, **odd**, or **neither**.

Even function	**Graphical definition:** A relation that has symmetry with respect to (w.r.t) the vertical axis. 	**Algebraic definition:** A relation that satisfies: $f(x) = f(-x)$
Odd function	**Graphical definition:** A relation that has symmetry with respect to (w.r.t.) the origin. 	**Algebraic definition:** A relation that satisfies: $f(x) = -f(-x)$

In the following examples, is the relation an even or an odd function (or neither)?

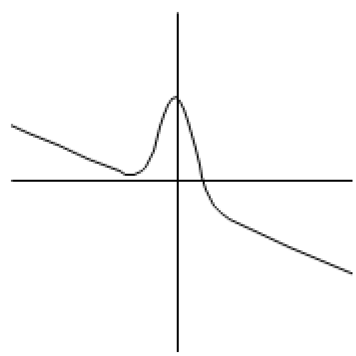
Example 1:

even



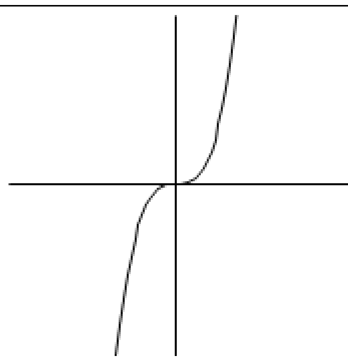
Example 2:

Neither



Example 3:

odd



Example 4: $f(x) = x^2 + 3$

Even test: $f(x) = f(-x)$

$x^2+3 = (-x)^2+3$

$x^2+3 \stackrel{\checkmark}{=} x^2+3$

$\boxed{\text{Even}}$

Example 5: $f(x) = 4x^3 - x$

Odd test: $f(x) = -f(-x)$

$4x^3 - x = -(4(-x)^3 - (-x))$

$4x^3 - x = -(-4x^3 + x)$

$4x^3 - x \stackrel{\checkmark}{=} 4x^3 - x$

$\boxed{\text{odd}}$

Example 6: $f(x) = x^2 + x^3 - 1$

Odd test: $f(x) = -f(-x)$

$x^2 + x^3 - 1 = -((-x)^2 + (-x)^3 - 1)$

$= -(x^2 - x^3 - 1)$

$x^2 + x^3 - 1 \neq -x^2 + x^3 + 1$

Even test: $f(x) = f(-x)$

$x^2 + x^3 - 1 = (-x)^2 + (-x)^3 - 1$

$x^2 + x^3 - 1 \neq x^2 - x^3 - 1$

$\boxed{\text{Neither}}$

Example 7: Complete the table to determine if $f(x) = x^2 + 1$ is even or odd function (or neither). The domain is restricted to {1, 2, 3, –1, –2}.

x	f(x)	f(-x)	-f(-x)
1	**2**	**2**	**–2**
2	**5**	**5**	**–5**
3	**10**	**10**	**–10**
–1	**2**	**2**	**–2**
–2	**5**	**5**	**–5**

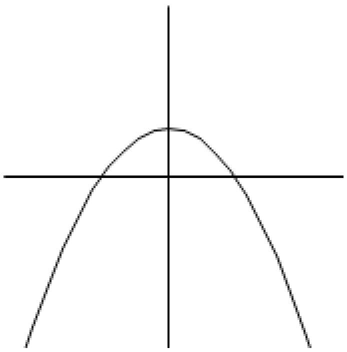
Even
The f(x) and f(-x) columns are identical

It is possible for a relation to have **symmetry w.r.t. the horizontal** (x) axis. It is a moot point to ask if this is an even or odd function because it is **not a function**…notice that it fails the vertical line test.

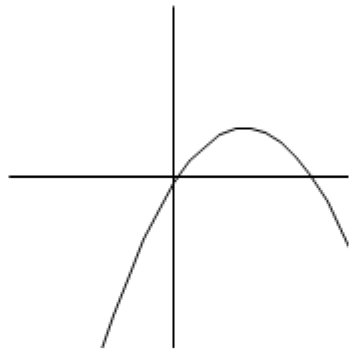
See **Calculator Appendix G** for a look at even and odd functions on the graphing calculator.

Assignment: In the problems 1-6, decide if the relation is an even or an odd function (or neither).

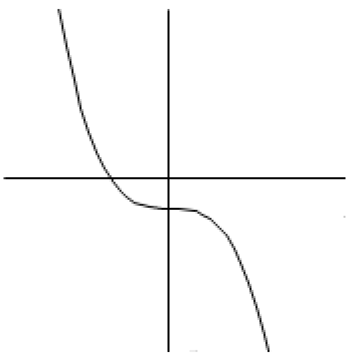
1.



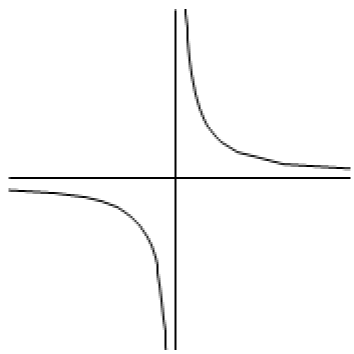
2.



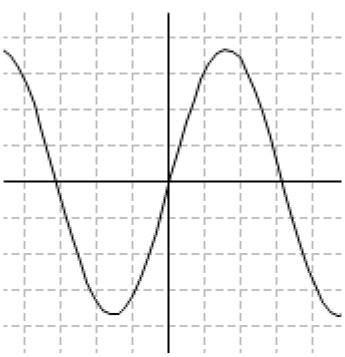
3.



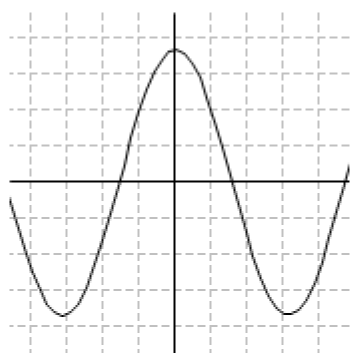
4.



5.



6.



7. Graphed even functions have symmetry w.r.t. what?

8. Graphed odd functions have symmetry w.r.t. what?

In problems 9-14, apply the algebraic tests to determine if the function is even, odd, or neither:

9. $f(x) = x/(x^2 - 1)$

10. $f(x) = x^2 - 2x$

11. $f(x) = |x|$

12. $f(x) = x^4 - x^2$

13. $f(x) = \sqrt[3]{x}$

14. $f(x) = \sqrt[3]{x} + 1$

Use the tables in problems 15-17 to determine if the function f(x) is even, odd, or neither.

15.

x	**f(x)**	**f(–x)**	**–f(–x)**	
1	4	–4	4	
2	8	–8	8	
3	32	–32	32	
–1	–18	18	–18	
–2	–20	20	–20	

16.

x	**f(x)**	**f(–x)**	**–f(–x)**	
1	4	4	–4	
2	8	8	–8	
3	–32	32	–32	
–1	–18	–18	18	
–2	–20	–20	20	

17.

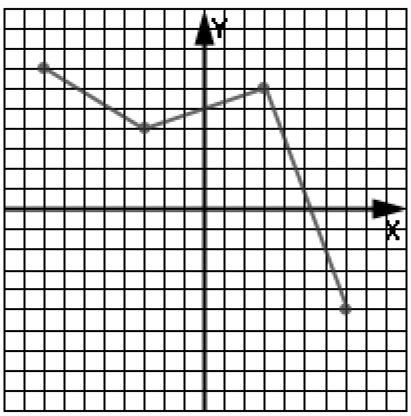
x	**f(x)**	**f(–x)**	**–f(–x)**	
1	4	4	–4	
2	8	8	–8	
3	32	32	–32	
–1	–18	–18	18	
–2	–20	–20	20	

Unit 6: Lesson 06 Transformations

Transformations of a function f(x) include:

- multiplying the function by a constant (neg or pos)
- adding a constant (neg or pos) to the function, and
- multiplying the independent variable (x) by a constant (neg or pos).

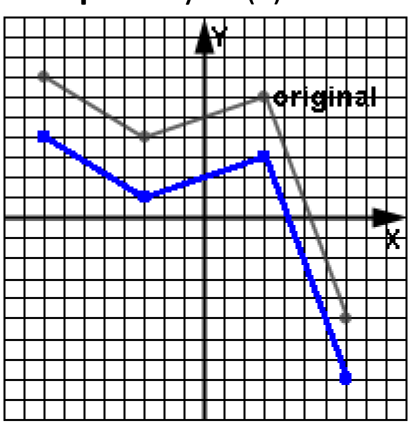
We begin with this function upon which transformations will be done. The corner and end points are shown in the table to the right.



x	f(x)
– 8	7
– 3	4
3	6
7	– 5

For each example, graph the transformation. For convenience, the initial function to be transformed is shown.

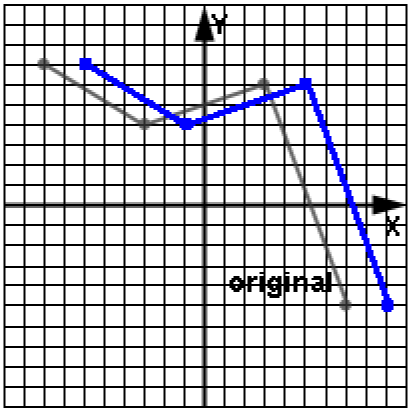
Example 1: y = f(x) – 3



Each point of the new function is the corresponding point of the original function **translated (shifted) down 3 units**.

Similarly, y = f(x) **+ 3** would produce a new function **shifted up 3 units**.

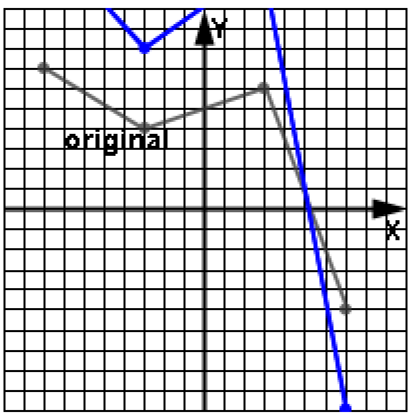
Example 2: y = f(x – 2)



Each point of the new function is the corresponding point of the original function **shifted right 2 units**.

Similarly, y = f(x **+ 2**) would produce a new function **translated (shifted) left 2 units**.

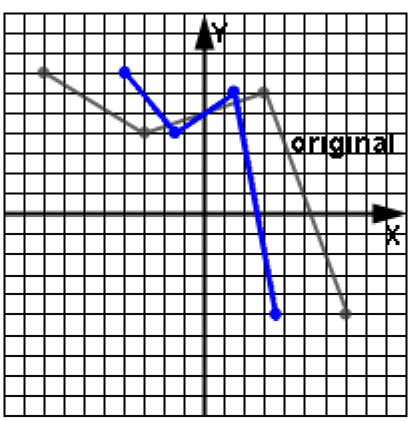
Example 3: y = 2f(x)



The **y-value** of each point of the new function i**s 2 times** the y-value of the corresponding point of the original function. The function has been **stretched in the <u>vertical</u> direction by a factor of 2**.

Similarly, y = (1/2)f(x) would produce a new function **that is <u>vertically</u> shrunk by a factor of 2**.

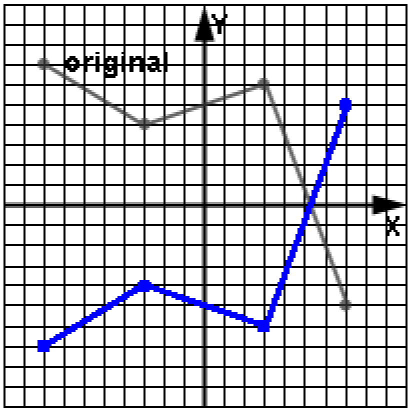
Example 4: y = f(2x)



The new function is **<u>horizontally</u> shrunk by a factor of 2**. (In order to maintain the former y-value of each point, the x-value must be half as much; thus, the shrinkage.)

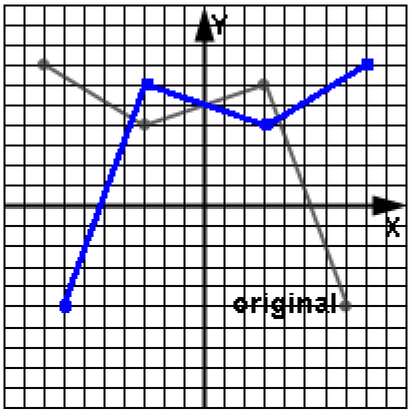
Similarly, y = f(x/2) **<u>horizontally</u> stretches by a factor of 2**.

Example 5: $y = -f(x)$



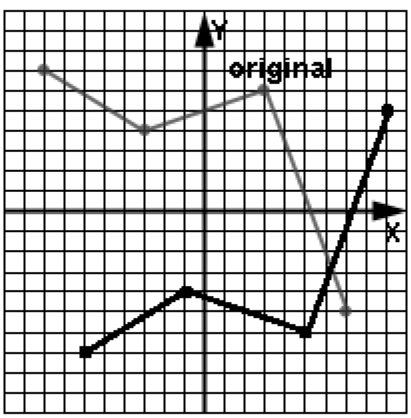
Each y-value of the new function is the negative of the corresponding y-value of the original function. The new function is the **reflection** of the original **across the x-axis**.

Example 6: $y = f(-x)$



Each x-value of the new function is the negative of the corresponding x-value of the original function. The new function is the **reflection** of the original **across the y-axis**.

Example 7: Give the function that describes the transformation from the original lighter graph (f(x)) to the darker one. Describe the transformation in words.

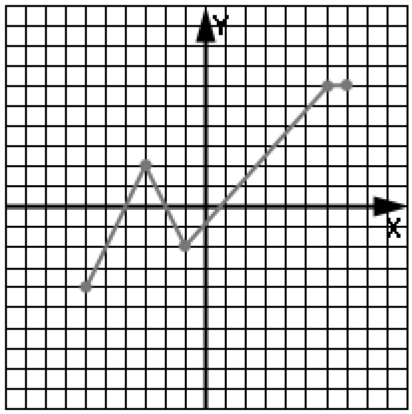


$-f(x-2)$

reflected across the x-axis, shifted right 2 units.

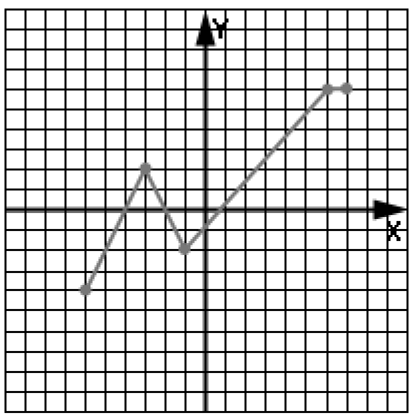
Assignment:

In problems 1- 6, begin with the function shown below. Draw the transformed function according to the given transformation and describe the transformation in words. (The corner and end points of this original function are shown in the table to the right.)

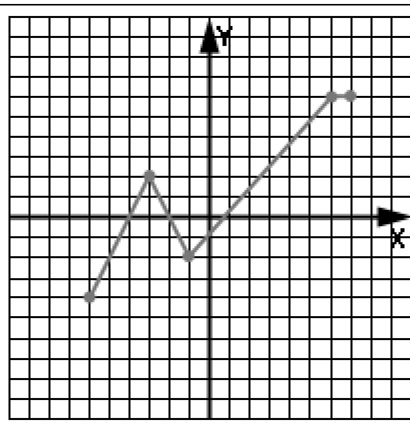


x	f(x)
– 6	–4
– 3	2
–1	–2
6	6
7	6

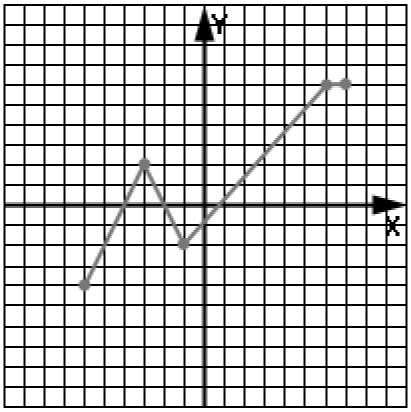
1. f(x) + 3



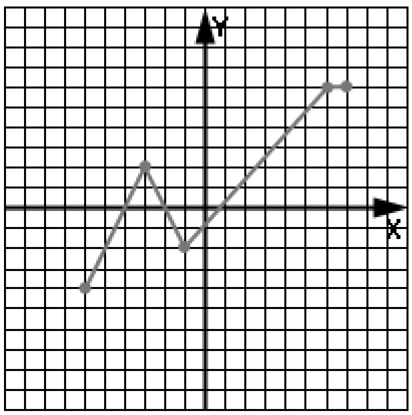
2. f(x + 3)



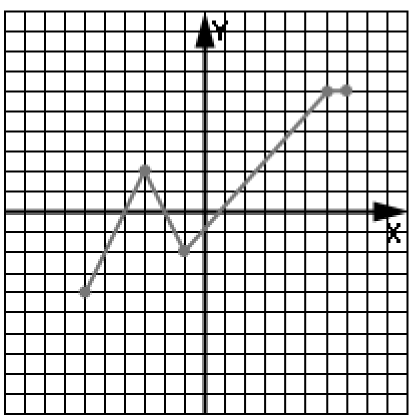
3. 2f(x)



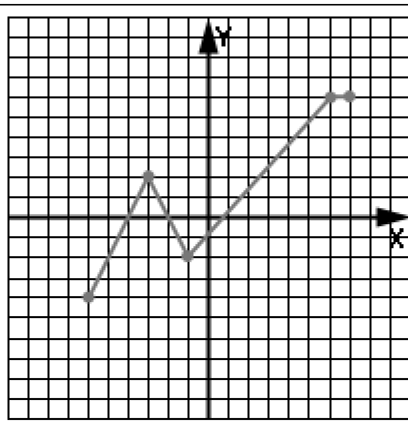
4. f(2x)



5. –f(x)

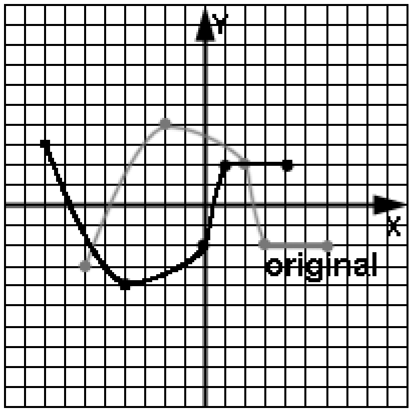


6. f(–x)

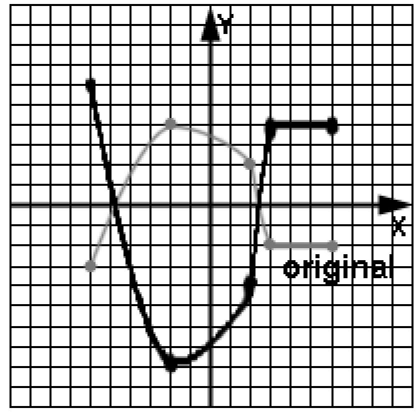


In problems 7-12, give the functional transformation from the lighter to the darker function and the word description of the transformation.

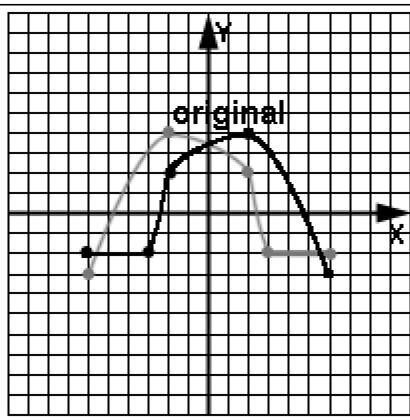
7.



8.



9.



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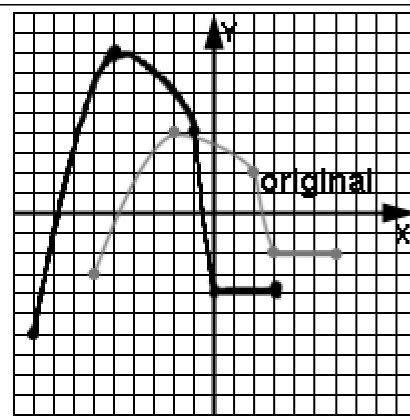

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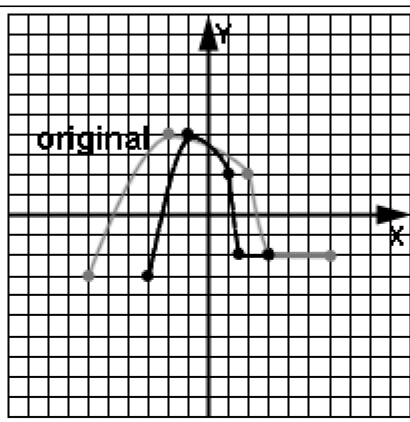

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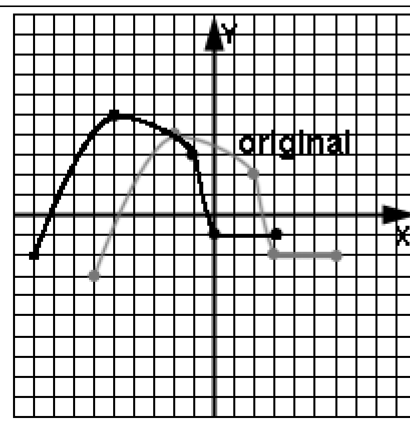
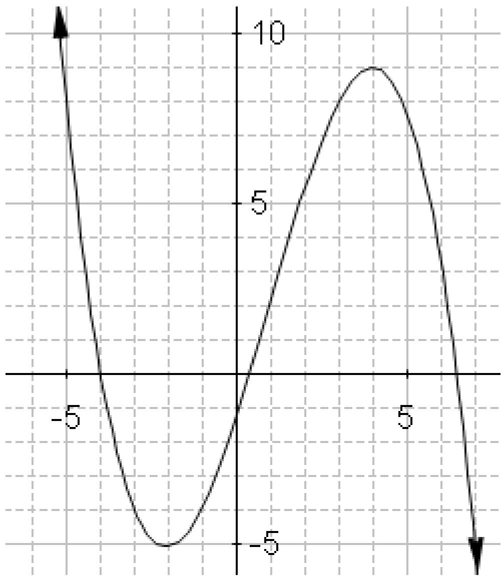


Unit 6: Lesson 07 Minimum and maximum



Specify the interval(s) where this function is increasing:

$(-2, 4)$

Specify the interval(s) where this function is decreasing:

$(-\infty, -2), (4, \infty)$

Maximum: Since f(x) above is **increasing to the left** of x = 4 and **decreasing to the right** of x = 4, f(x) has a **local maximum** at x = 4.

What is the local maximum function value? 9

Notice that a local maximum is not necessarily an **absolute maximum**. On the left side of this particular graph, the function continues to go higher without any limit. So there is no absolute maximum.

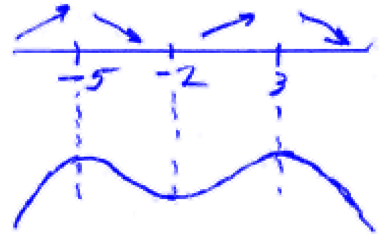
Minimum: Since f(x) above is **decreasing to the left** of x = –2 and **increasing to the right** of x = –2, f(x) has a **local minimum** at x = –2.

What is the local minimum function value? -5

Notice that a local minimum is not necessarily an **absolute minimum**. On the right side of this particular graph, the function continues to go lower without any limit. So there is no absolute minimum.

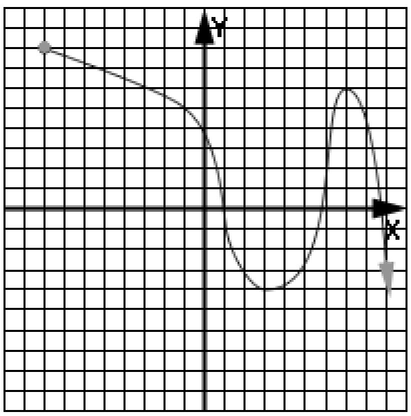
Example 1: Consider the function f(x) where increasing intervals are given by (–∞,–5) and (–2, 3). Decreasing intervals are given by (–5, –2) and (3, ∞).

Roughly sketch the function and give the x-values of any local maxima and/or minima.



max at $x=-5, 3$ min at $x=-2$

Example 2:



What are the locations of the local maxima?

$x=7$

Justify the above answer.

f increases to left of 7, & decreases to the right

What is the local function maximum?

$y=6$

What are the location(s) of the local minima?

$x=3$

Justify the above answer.

decreasing to the left of 3, increasing on the right.

What is the local function minimum?

$y=-4$

What is the absolute maximum function value?

$y=8$

What is the absolute minimum function value?

None, it goes to $-\infty$

Give the intervals over which the function is decreasing:

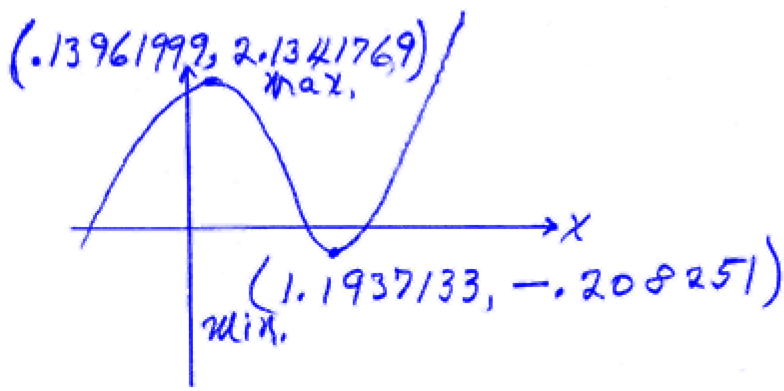
$(-8,3), (7,\infty)$

Give the intervals over which the function is increasing:

$(3,7)$

See **Calculator Appendix J** for how to use a graphing calculator to find the minimum (minima) and maximum (maxima) points of a function.

Example 3: Use a graphing calculator to find the minima and maxima of the function $f(x) = 4x^3 - 8x^2 + 2x + 2$. Make a sketch of the graphed function.

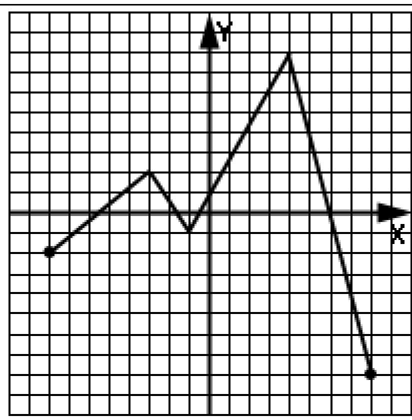


Assignment:

1. Consider the function f(x) where the only increasing interval is given by (–8,7). The only decreasing interval is given by (7, 9).

Roughly sketch the function and give the x-values of any local maxima and/or minima.

2.



What are the locations of the local maxima?

Justify the above answer.

What are the local maximum values?

Give the interval(s) over which the function is decreasing:

What are the locations of the local minima?

Justify the above answer.

Give the interval(s) over which the function is increasing:

What are the local minimum value(s)?

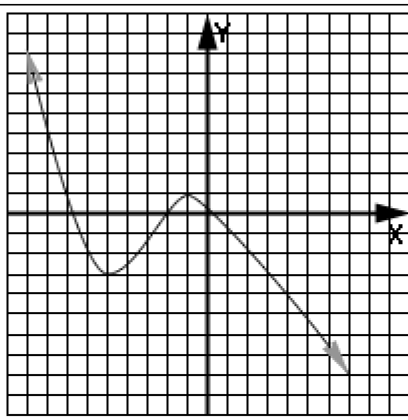
What is the absolute maximum?

What is the absolute minimum?

3. Consider the function f(x) where increasing intervals are given by (–∞,–1), (2, 5), and (8, 12). Decreasing intervals are given by (–1, 2) and (5, 8).

Roughly sketch the function and give the x-values of any local maxima and/or minima.

4.



What are the locations of the local maxima?

Justify the above answer.

What are the local maxima?

Give the interval(s) over which the function is decreasing:

What are the locations of the local minima?

Justify the above answer.

Give the interval(s) over which the function is increasing:

What are the local minima?

What is the absolute maximum?

What is the absolute minimum?

5. Use a graphing calculator to find the local minima and local maxima of the function $f(x) = -2x^3 + 5x^2 - 2x - 2$. Make a sketch of the graphed function.

Unit 6: Cumulative Review

1. Use a graphing calculator to perform a linear regression on the provided data and show the equation of the best-fit line along with a sketch of the line and scatter-plot. See **Calculator Appendices M and N** for a review.

x	y
–4.0	9.0
–1.1	4.2
2.0	–0.5
1.9	–2.0
6.0	–5.0
5.8	–7.0

2. What is the equation of the line that is parallel to the y-axis and has –5 as a root?

3. Give the definitions of all six trig functions in terms of x, y, and r.

4. Solve the system $3x - 2y = 5$, $4x + y = 1$ by the elimination method.

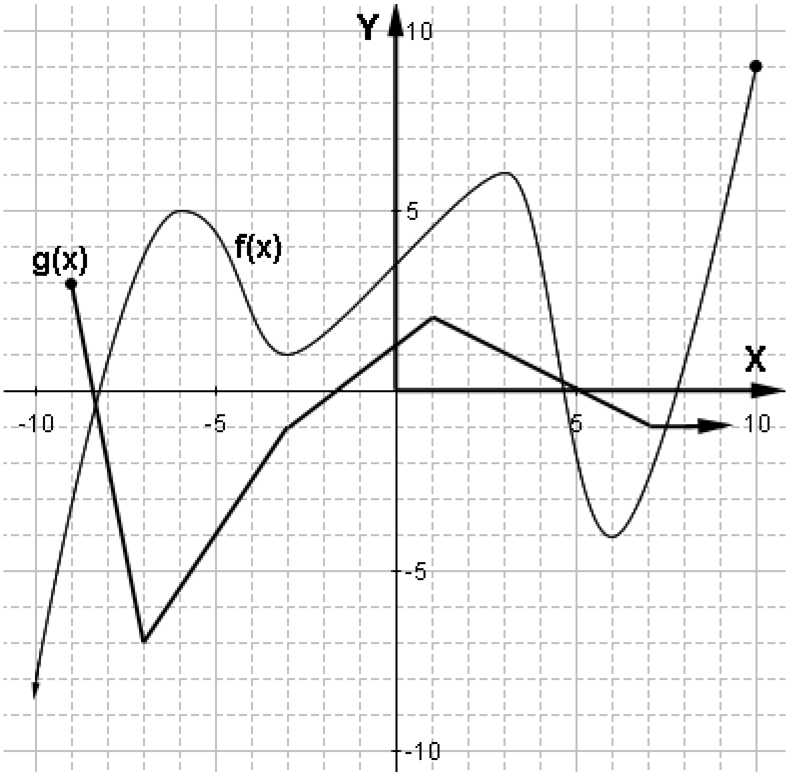
5. Solve $4\tan(x) - \cot(x) = 2$. Limit answers to the 1st & 4th quadrants.

6. An airplane flies at 260 mph with a navigational heading of 76^{o}. The wind is blowing due west at 52mph. What is the ground speed and navigational heading of the ground track?

7. If $\tan A = 2$ and $\tan B = 1.4$ what is the numerical value of $\tan(A + B)$?

8. Solve the triangle where A= 19^o, b = 4, and C = 130^o.

Unit 6: Review



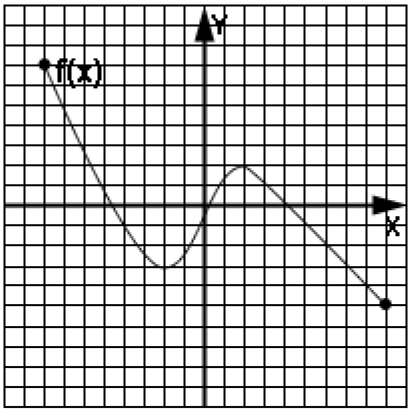
Use these two functions in problems 1-18 to find the indicated values or intervals.

1. f(9)	2. g(8)	3. g(f(8))
4. Absolute maximum for f(x)	5. Absolute minimum for f(x)	6. Domain for f(x)
7. Range for f(x)	8. Domain for g(x)	9. Range for g(x)

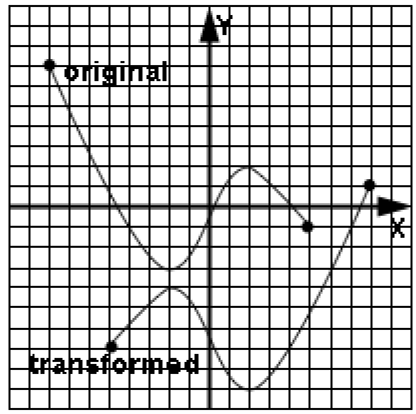
10. Location of local maxima for f(x)	11. Location of local minima for g(x)	12. Interval(s) where f(x) increases
13. Interval(s) where g(x) decreases	14. (f + g)(-3)	15. Roots of f(x)
16. y-intercept of f(x)	17. Is f(x) a function?	18. Is g(x) one-to-one?
19. Draw an example of an even function.	20. Draw an example of an odd function.	21. What is the algebraic test that determines if f(x) is odd?

22. With algebraic tests, determine if $y = 4x^6 + 3x^4$ is even, odd, or neither.

23. The function f(x) is shown here. Draw the reflection of f(x) across the x-axis. How is this new function represented in terms of f(x)?



24. The original f(x) is shown here along with a transformation of it. How is this new function represented in terms of f(x)? Describe the transformation in words.

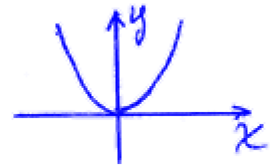


Pre Calculus, Unit 7

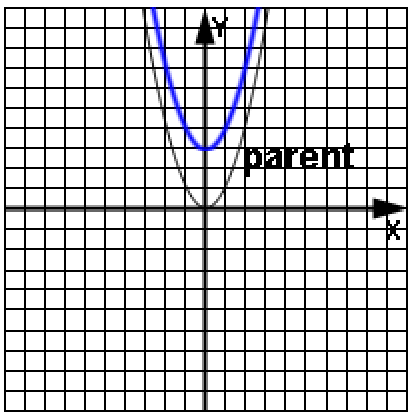
Quadratic Functions (parabolas)

Unit 7: Lesson 01

Transformations of quadratic functions

The quadratic parent function graphs as a parabola:	$y = f(x) = x^2$ 

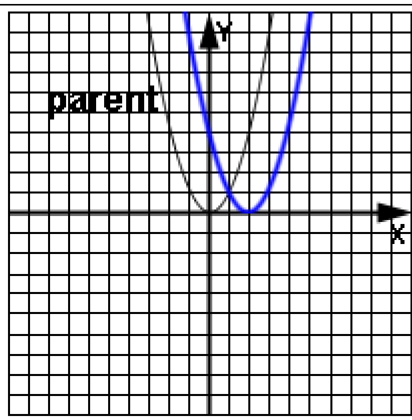
In the examples 1-6, begin with this parent function $y = f(x) = x^2$ and graph the indicated transformation:



Example 1: $y = f(x) + 3$

Each point of the new function is the corresponding point of the original parent function **translated (shifted) up 3 units**.

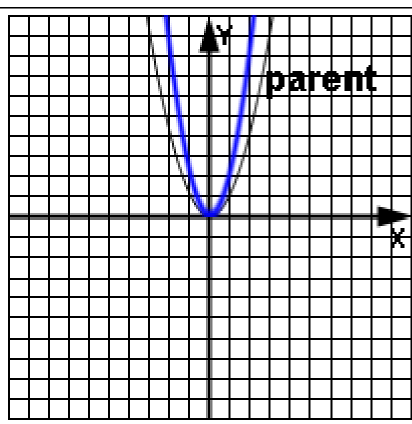
Similarly, y = f(x) **– 3** would produce a new parabola **shifted down 3 units**.



Example 2: $y = f(x - 2)$

Each point of the new function is the corresponding point of the original parent function **translated (shifted) right 2 units**.

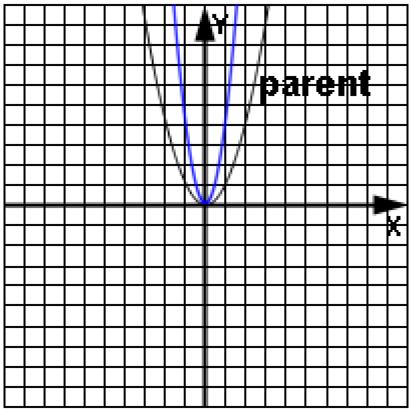
Similarly, $y = f(x + 2)$ would produce a new parabola **shifted left 2 units**.



Example 3: $y = 2f(x)$

The **y-value** of each point of the new function **is 2 times** the y-value of the corresponding point of the original parent function. The function has been **stretched in the <u>vertical</u> direction by a factor of 2**.

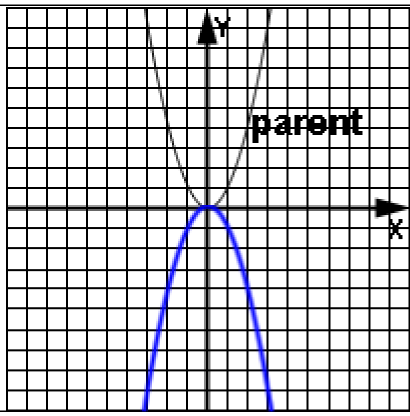
Similarly, $y = (1/2)f(x)$ would produce a new function **that is <u>vertically</u> shrunk by a factor of 2.**



Example 4: y = f(2x)

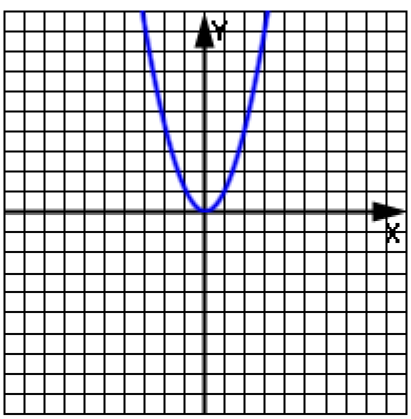
The new function is **<u>horizontally</u> shrunk by a factor of 2**. (In order to maintain the former y value of each point, the x value must be **half** as much; thus, the shrinkage.)

Similarly, y = f(.5x) **<u>horizontally</u> stretches by a factor of 2**.



Example 5: y = –f(x)

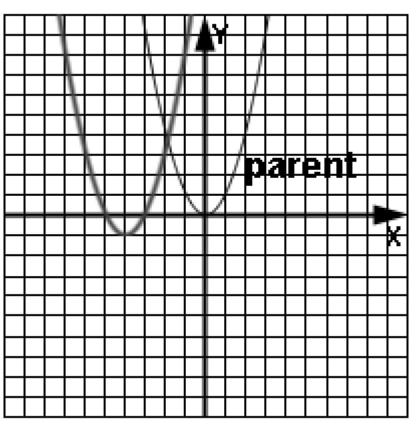
Each y-value of the new function is the negative of the corresponding y-value of the original parent function. The new function is the **reflection** of the original **across the x-axis**.



Example 6: y = f(–x)

Each x-value of the new function is the negative of the corresponding x-value of the original parent function. The new function is the **reflection** of the original **across the y-axis**.

Notice that this reflection reproduces itself since it is an even function.



Example 7:

Give the function that describes the transformation from the original lighter parent (f(x)) to the darker function. Describe the transformation in words.

$f(x+4)-1$

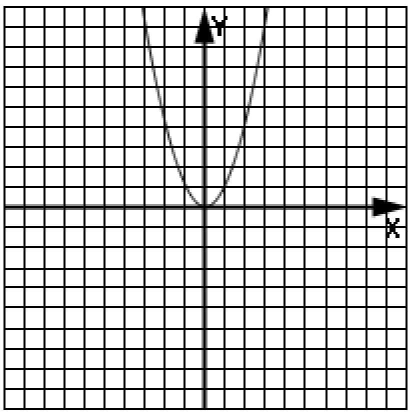
shift left 4, down 1

Example 8: Given the transformation $4f(x–2) + 8$, write the equation for y strictly in terms of x where the original parent function is $f(x) = x^2$.

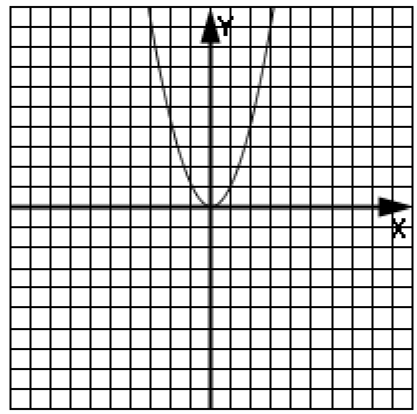
$$y = 4(x-2)^2 + 8$$

Assignment: In problems 1- 6, graph the transformed parent function f(x). Describe the transformation in words.

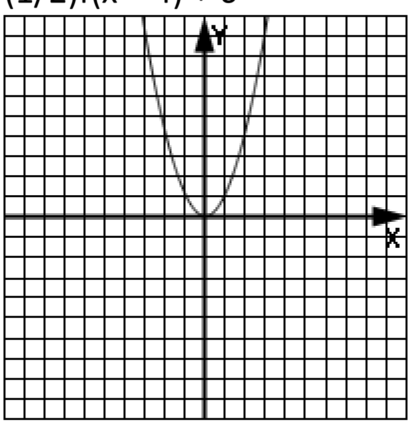
1. f(x – 1) + 3



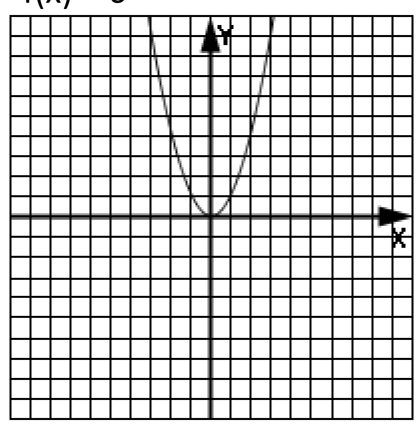
2. –f(x) – 2



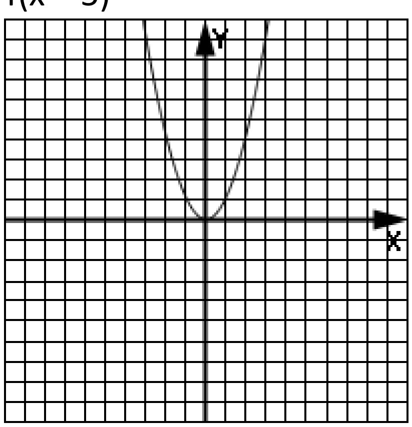
3. (1/2)f(x – 4) + 6



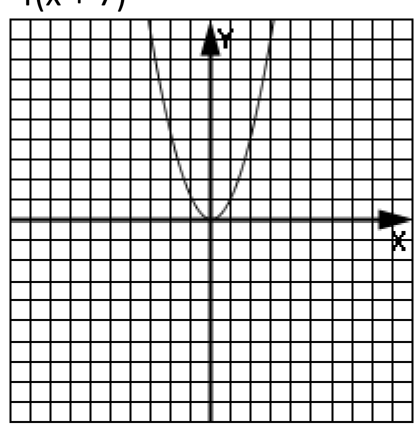
4. –f(x) – 6



5. f(x – 5)

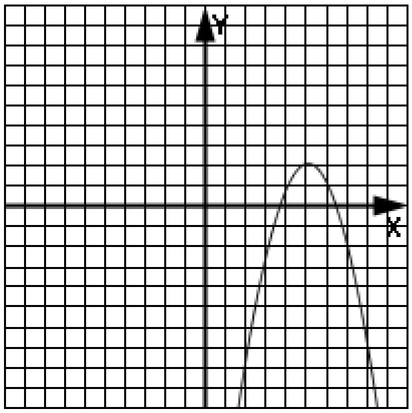


6. –f(x + 7)



In problems 7-12, give the transformation in terms of *f* that transforms the original parent function (y = f(x) = x^2) to the graphs shown. Give a word description of the transformation.

7.



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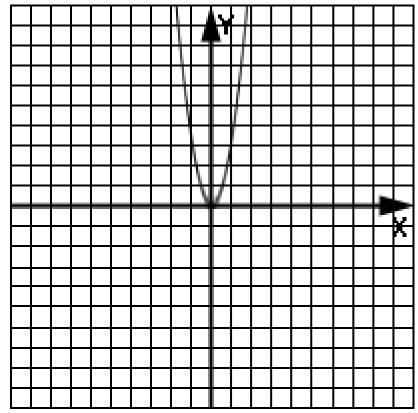

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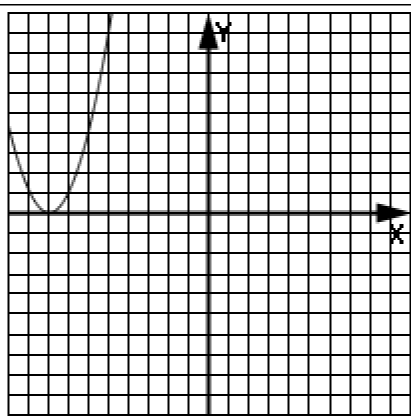

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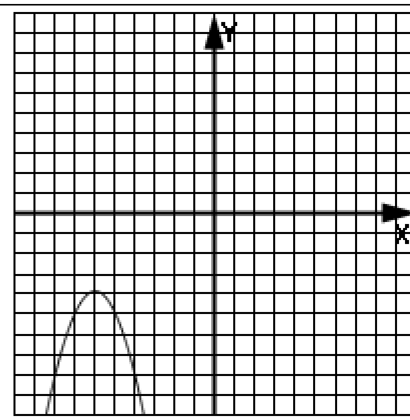

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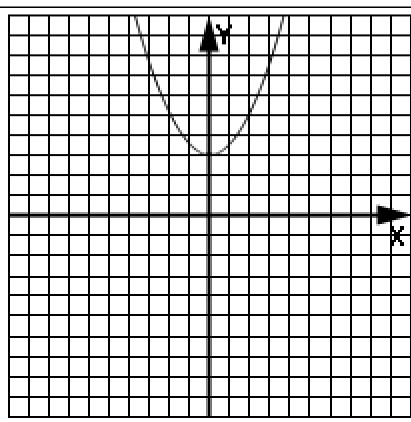

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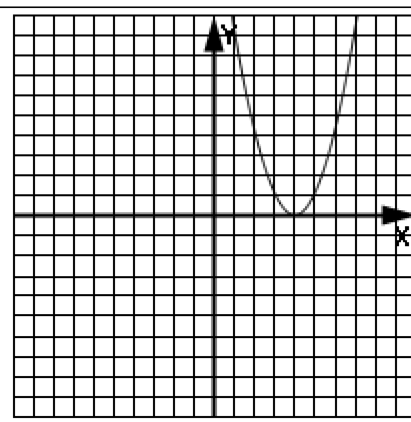


13. Given the transformation $-3f(x + 5) - 6$, write the equation for y strictly in terms of x where the original parent function $f(x) = x^2$.

14. Given the transformation described by, “reflect across the x-axis, shift right 7 and down 2”, write the equation for y strictly in terms of x where the original parent function $f(x) = x^2$.

Unit 7: Lesson 02 Three forms of the quadratic function

Standard-form:

$f(x) = ax^2 + bx + c$

Vertex at $(h, k) = \left(-\frac{b}{2a},\ f\left(-\frac{b}{2a}\right)\right)$

(See **Enrichment Topic N** for how we get this.)

Roots given by the quadratic formula

Axis of symmetry at $x = \frac{-b}{2a}$

y-intercept at c

Vertex-form:

$f(x) = a(x - h)^2 + k$

Vertex at (h, k)

Axis of symmetry at x = h

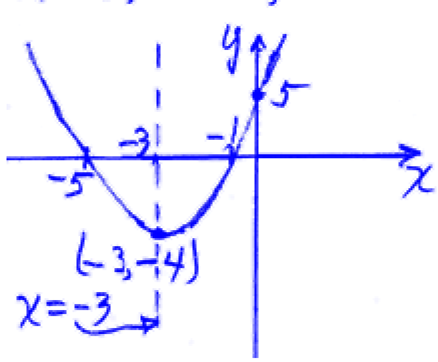
Root-form:

$f(x) = a(x - r_1)(x - r_2)$

r_1 and r_2 are the roots

Example 1: For the standard-form function, $f(x) = x^2 + 6x + 5$, find the vertex, roots, axis of symmetry, and the y-intercept. Sketch a graph of the function and label completely.

$a = 1;\ b = 6;\ c = 5$



$h = \frac{-b}{2a} = \frac{-6}{2 \cdot 1} = -3$

$k = f(-3) = (-3)^2 + 6(-3) + 5$

$= 9 - 18 + 5 = -4$

y-int $= c = 5$

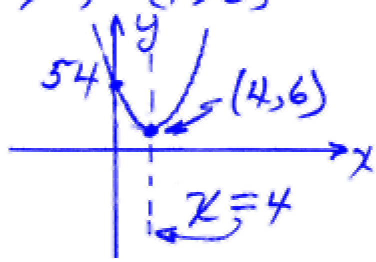
roots

$x = \frac{-6 \pm \sqrt{36 - 4 \cdot 1 \cdot 5}}{2 \cdot 1}$

$x = \frac{-6 \pm 4}{2} = -1, -5$

Example 2: For the vertex-form function, $f(x) = 3(x - 4)^2 + 6$, find the vertex, roots, axis of symmetry, and the y-intercept. Sketch a graph of the function and label completely.

$(h, k) = (4, 6)$



roots: $0 = 3(x-4)^2 + 6$

$(x-4)^2 = -2$

$x - 4 = \pm\sqrt{-2}$

complex roots

y-int: $y = 3(0-4)^2 + 6$

$y = 3(16) + 6 = 54$

Example 3: For the root-form function, $f(x) = 3(x - 10)(x + 6)$, find the vertex, roots, axis of symmetry, and the y-intercept. Sketch a graph of the function and label completely.

$f(x) = 3(x-10)(x-(-6))$

$r_1 = 10,\ r_2 = -6$

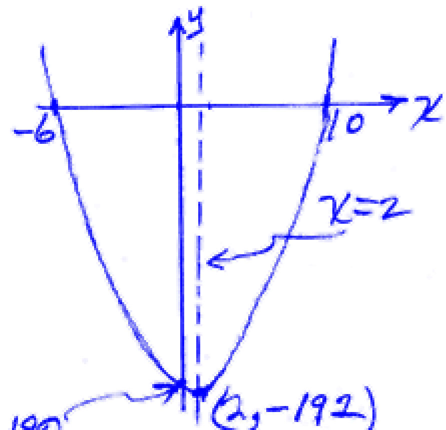
h is half way between

$h = \frac{r_1 + r_2}{2} = \frac{10-6}{2} = 2$

$k = f(2) = 3(2-10)(2+6)$

$= 3(-8)8 = -192$

y-int $\rightarrow$ $y = 3(0-10)(0+6) = -180$



Example 4: Write the equation of parabola that has a maximum point at $f(2) = -3$ and passes through $(6, -8)$.

$(h, k) = (2, -3)$

$y = a(x-h)^2 + k$

$y = a(x-2)^2 - 3$

Now sub in $(6, -8)$

$-8 = a(6-2)^2 - 3$

$a(4)^2 - 3 = -8$

$16a = -8 + 3$

$a = -\frac{5}{16}$

$y = a(x-h)^2 + k$

$\boxed{y = -\frac{5}{16}(x-2)^2 - 3}$

Example 5: Write the equation of the quadratic function that has roots at (–4, 0) and (2, 0) and a minimum value of –10.

$r_1 = -4;\ r_2 = 2$

$y = a(x - r_1)(x - r_2)$

$y = a(x - (-4))(x - 2)$

$y = a(x+4)(x-2)$

sub in $(-1, -10)$

$-10 = a(-1+4)(-1-2)$

$a = -10/(3\cdot(-3)) = 10/9$

h = halfway between the roots

$h = \frac{-4+2}{2} = -1$

$k = -10$ vertex $\rightarrow (-1, -10)$

$y = a(x - r_1)(x - r_2)$

$$\boxed{y = \frac{10}{9}(x+4)(x-2)}$$

Assignment:

1. For the function, $f(x) = x^2 + 4x – 60$, find the vertex, roots, axis of symmetry, and the y-intercept. Sketch a graph of the function and label completely.

2. For the function, $f(x) = 3(x – 1)^2 – 6$, find the vertex, roots, axis of symmetry, and the y-intercept. Sketch a graph of the function and label completely.

3. For the function, $f(x) = 4(x - 5)(x + 2)$, find the vertex, roots, axis of symmetry, and the y-intercept. Sketch a graph of the function and label completely.

In problems 4-8, use the given information to produce a quadratic function in a form that is best suited to the problem.

---

4. The vertex is located at $(4, -1)$ and the parabola passes through $(2, 3)$.

5. The quadratic function passes through the origin and has its vertex at (5, 8).

6. The parabola's y-intercept is at (0, 12) and the x-intercepts are at (–1, 0) and (7, 0).

7. The quadratic function has a point given by f(0) = 1 and zeros at (6, 0) and (2, 0).

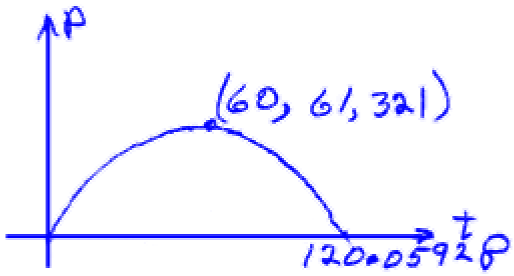
8. The parabola passes through the point given by $f(3) = 1$ and has a maximum value given by $f(-2) = 8$.

Unit 7: Lesson 03 **Quadratic calculator applications**

Example 1: A beekeeper's hives are making honey at a constant rate; yet the price of honey is going steadily down. Let t = the time in days from the start of the honey season. The profit from honey sales as a function of time is given by:

$$P(t) = -17t^2 + 2040t + 121$$

(a)Use a graphing calculator to graph the function.

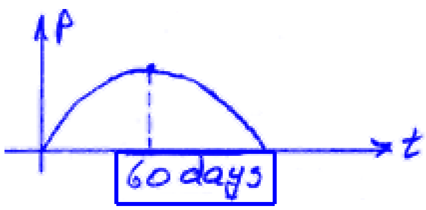


(b) Find the domain (find the roots) and range (find the maximum point).

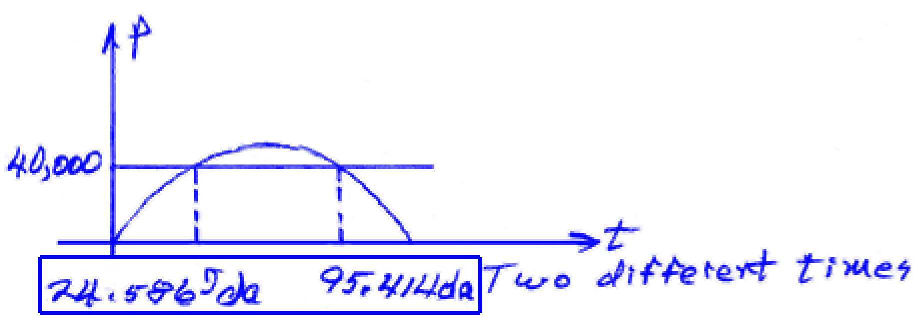
Domain: [0, 120.05928]

Range: [0, 61,321]

(c) After how many days should he harvest his honey in order to realize a maximum profit?



(d) When could he harvest to realize a profit of $40,000?



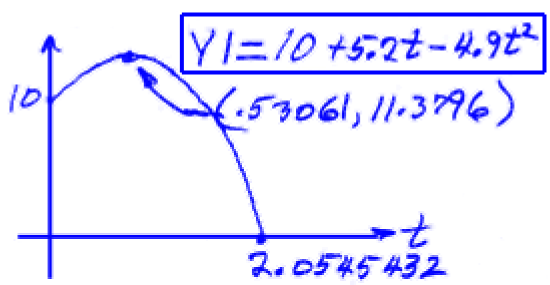
Consider an object that only moves vertically under the influence of gravity. The function of time that describes the position above the ground is:

$$y(t) = y_o + v_o t - 4.9t^2 \quad \text{(distance is in meters, time is in seconds)}$$

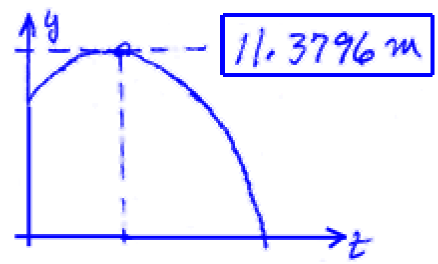
y_o is the initial position (at t = 0) of the object above ground
v_o is the initial upward speed (at t = 0) of the object

Example 2: A man at the top and near the edge of 10 m tall building throws a ball vertically upward with a speed of 5.2 m/sec.

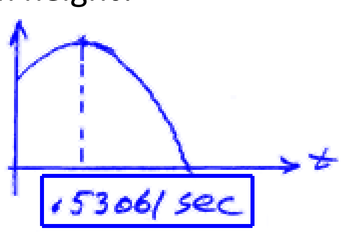
(a) Specify the function of *t* describing the vertical position of the ball and then graph it.



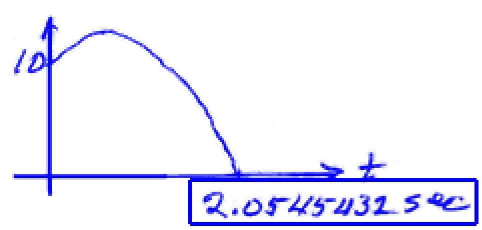
(b) What is the maximum height above ground that the ball reaches?



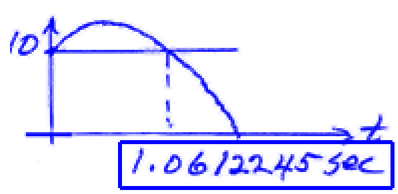
(c) When does the ball reach the maximum height?



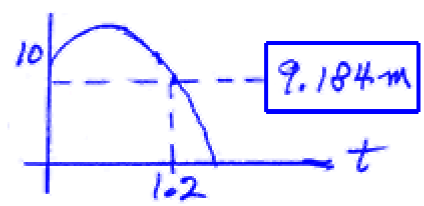
(d) When does the ball reach the ground?



(e) When does the ball pass the top of the roof of the building on the way down?



(f) How high is the ball after falling 1.2 sec?



The following reviews may prove useful in working the problems in this lesson:

See **Calculator Appendix C** for how to find intersection points of curves.

See **Calculator Appendix I** for how to find roots.

See **Calculator Appendix J** for how to find maximum points.

Assignment:

1. The path followed by the baseball in a home-run by Babe Ruth was given by:
$f(x) = -.0029x^2 + x + 3.1$ (all dimensions are in feet)

(a) Graph the function.	(b) Give the domain and range of the function.
(c) What is greatest height of the ball?	(d) How far from home plate did the ball fall?
(e) If there was a 15 ft fence at 335 ft from home plate, would the ball clear the fence?	(f) At what points is the ball 50 ft above the ground?

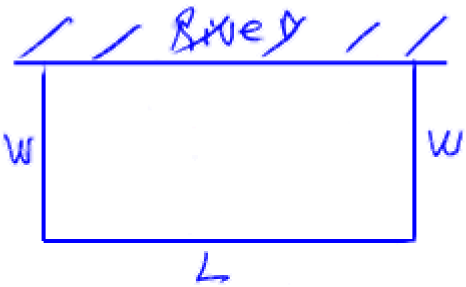
2. In a Myth Busters Experiment, Jamie is on the 4th floor (20 m above the ground). Adam is on the 5th floor (25 m above the ground). Simultaneously, Jamie releases a rock and Adam throws his rock downward with a speed of 5 m/sec.

(a) Give the position of each person's rock as a function of time.	(b) On the same coordinate system, graph the vertical positions of both rocks above ground as functions of time.
(c) When does Adams's rock pass the 4th floor?	(d) When are the two rocks at the same height?
(e) When is Jamie's rock 10 m above the ground?	(f) How much later after Adam's rock strikes the ground does Jamie's rock reach the ground?

Unit 7: Lesson 04 Quadratic area applications

Example: A rectangular cotton field is to be enclosed by a fence alongside a river. The river serves as one side of the field while the fence serves as the boundaries of the other three sides. The farmer has a total of only 1000 meters of fence available.

(a) Draw the field and label the *L* and *W* sides.



(b) Write an equation in which all three sides are summed and set equal to 1000. Solve for the *L* variable.

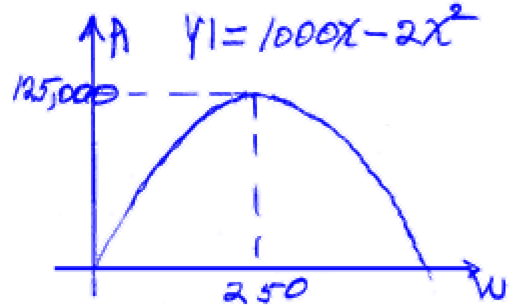
$$W + W + L = 1000$$
$$2W + L = 1000$$
$$L = 1000 - 2W$$

(c) Substitute from the equation in part (b) into the equation for the area of the field so as to produce area as a function of the width.

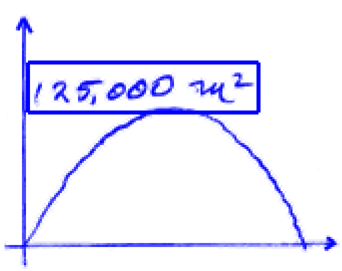
$$A = W(L)$$
$$A = W(1000 - 2W)$$
$$A = \boxed{1000W - 2W^2}$$

(d) Use a graphing calculator to graph the area of the field as a function of the width of the field.

$Y1 = 1000x - 2x^2$



(e) What is the maximum area?



(f) What should be the dimensions of the field so area is a maximum?

$$W = \boxed{250 \text{ m}}$$
$$2W + L = 1000$$
$$2(250) + L = 1000$$
$$\boxed{L = 500 \text{ m}}$$

Assignment:

1. A landowner wants to build a rectangular building that has the maximum area. Local building codes dictate that the perimeter be no more that 450 ft. What will be the maximum floor space that he can build? What would be the dimensions of the building?

2. Cowboy Copus intends to build a corral of maximum area. It has been a hard year and he can only afford to buy 320 ft of fence. One side of the corral is to have a 10 ft gap where he will eventually put a gate. What will be the maximum area he can enclose and what will be the dimensions of the rectangular corral?

3. A rectangular parcel of land is to be enclosed on all four sides by a fence. With two interior parallel cross-fences, it is to be subdivided into three separate areas. If only 800 ft of fence is available, what is the maximum area possible for the original rectangular parcel? What are its dimensions?

4. Harriet wants to create a small rectangular flower garden in her larger rectangular back yard. She wants to put it in a corner so that her yard fence serves as two of the sides of the garden. She has only 30 ft of edging to create the other two sides of the garden. In order to maximize the area of the flower garden, what should the dimensions be? What is the maximum area?

Unit 7: Cumulative Review

1. Find the equation of the line parallel to the line given by $5x - 2y = 11$ and passing through the intersection of two other lines. One of those intersecting lines is the vertical line given by $x = -5$ and the other is a horizontal line given by $y = 9$.

---

2. Algebraically determine if $y = 3x/(x - 2)^2$ is even, odd, or neither.

---

3. Solve the triangle ABC where $C = 35^o$, $a = 7$, $b = 15$.

4. Prove the identity $\frac{\sec A}{1-\cos A} = \frac{\sec A+1}{\sin^2 A}$.

5. Evaluate this determinant:
$\begin{vmatrix} 3 & -1 \\ 2 & 7 \end{vmatrix}$

6. Multiply these two matrices:
$\begin{bmatrix} 2 & -4 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 6 & 1 \\ 2 & 2 \end{bmatrix}$

7. Factor $x^4 - 16$ completely.

8. Solve $3x^2 - 7x + 2 = 0$ by factoring.

9. Solve $3x^2 - 7x + 2 = 0$ by completing the square.

10. Solve the system $3x - 2y = 5$; $4x + y = 1$ using the substitution method.

Unit 7: Review

1. Sketch the parabola $y = -f(x - 7) + 2$ where the original parent function is $f(x) = x^2$. Label the vertex, roots, y-intercept, and axis of symmetry.

---

2. Sketch the parabola $y = 4x^2 - 2x - 1$ labeling both the vertex and axis of symmetry.

---

3. Sketch the parabola $y = 6(x - 6)^2 + 4$ labeling both the vertex and axis of symmetry. Comment on the roots.

4. Sketch the parabola $y = 3(x – 1)(x – 4)$ labeling the roots, vertex, and axis of symmetry.

---

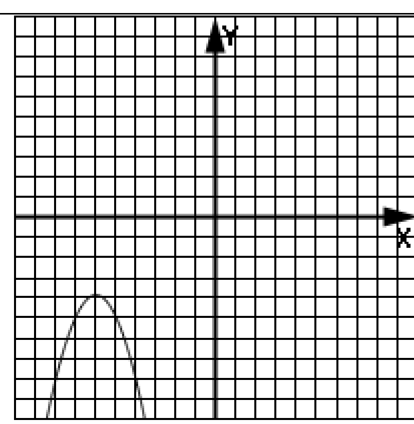
5. Write the equation of a quadratic function that has a minimum point at (5, –2) and passes through (7, 1).

---

6. Write the equation of a parabola that has a maximum value of 7 and with roots at –4 and 10.

---

7. Describe the transformation in terms of *f* of the function shown to the right. The original parent function was $f(x) = x^2$. Also describe the transformation in words.



8. A pig pen is to be built up against the long side of a barn with the barn serving as one side of the pen. There is a maximum of only 60 ft of fence material to make up the other three sides. The side of the pen opposite the barn is to have an 8 foot gap so as to accommodate a gate. What should be the dimensions of the pen in order to maximize its area? What is the maximum area?

9. An object is dropped from a hot-air balloon that is ascending at the rate of 2 m/sec when the balloon is 200 m above the ground. Write the function of time that describes the position of the object above the ground. Use a graphing calculator to graph this function and to determine when it strikes the ground. How high is the object 2.3 sec after it is released from the balloon?

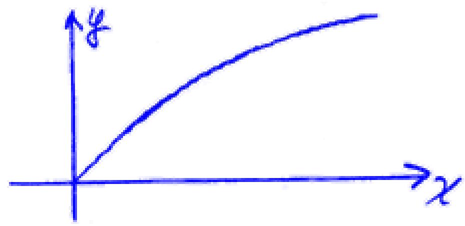
10. A salesman for a widget factory has determined that revenue for a day is dependent on the price of a widget according to $R = -2p^2 + 150p$ where p is the price of an individual widget. Use a graphing calculator to determine the optimum price of a widget so as to maximize revenue for a single day. What is the maximum possible revenue in a day?

Pre Calculus, Unit 8

Special functions

Unit 8: Lesson 01 Square root and semicircle functions

Consider the function f(x) = $\sqrt{x}$. Substitute in x values such as 4, 9, 16, etc., and we are quickly able to make a rough sketch.



x	y
0	0
1	1
4	2
9	3
16	4

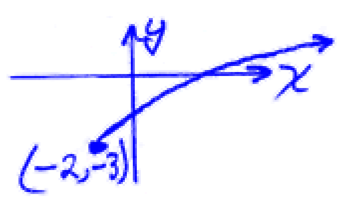
The characteristics of this function are:

Domain: $[0, \infty)$ Function? yes

Range: $[0, \infty)$ One-to-one? yes

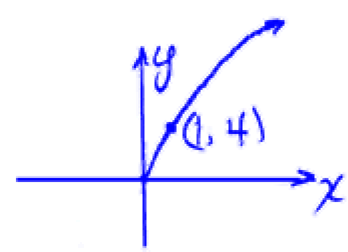
Apply our previous experience with transformations of functions to this new parent function. In the following examples graph the transformations of the original function, $\sqrt{x}$, give the domain and range, and give a word description of the transformation.

Example 1: $f(x) = \sqrt{x+2} - 3$



D: $[-2, \infty)$
R: $[-3, \infty)$
shift left 2, down 3

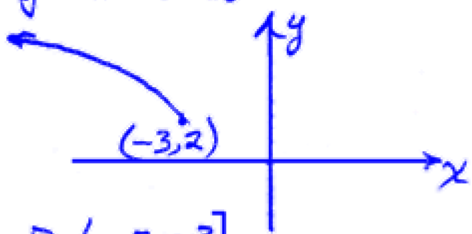
Example 2: $f(x) = 4\sqrt{x}$



D: $[0, \infty)$
R: $[0, \infty)$
vertically stretch by a factor of 4.

Example 3: $f(x) = \sqrt{-x-3} + 2$

$y = \sqrt{-(x+3)} + 2$



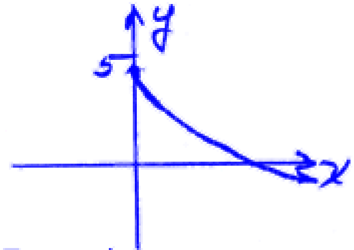
D: $(-\infty, -3]$

R: $[2, \infty)$

Reflect across the y-axis

shift left 3, up 2.

Example 4: $f(x) = -\sqrt{x} + 5$



D: $[0, \infty)$

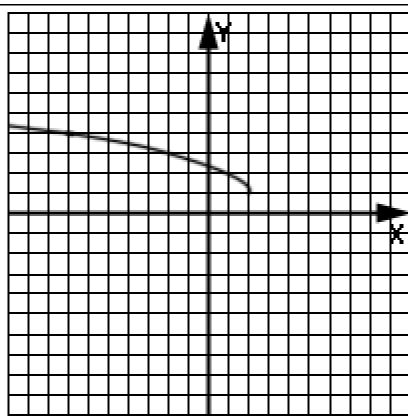
R: $(-\infty, 5]$

Reflect across the x-axis,

translate up 5.

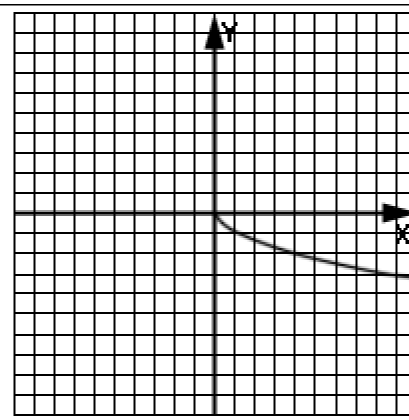
In the following two examples, write the function corresponding to the graph.

Example 5:



$f(x) = \sqrt{-(x-2)} + 1$

Example 6:



$f(x) = -\sqrt{x}$

Consider the equation of a circle centered at the origin and with radius r. Solve for y:

$x^2 + y^2 = r^2$

$y^2 = r^2 - x^2$

$y = \pm\sqrt{r^2 - x^2}$

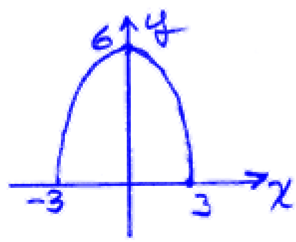
The positive solution, $\mathbf{y = f(x) = +\sqrt{r^2 - x^2}}$, produces the **top half** of a circle and is our new **semi-circle parent function**.

The negative solution, $\mathbf{y = f(x) = -\sqrt{r^2 - x^2}}$, produces the **bottom half** of a circle.

In examples 7 and 8, sketch the transformed semi-circle, give the domain and range, and give a word description of the transformation.

Example 7: $f(x) = 2\sqrt{9 - x^2}$

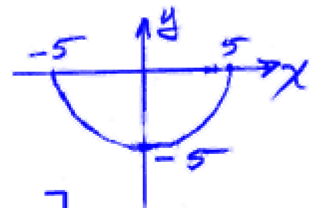
$r^2 = 9$
$r = 3$



D: [-3, 3]
R: [0, 6]
Vertically stretch by a factor of 2.

Example 8: $f(x) = -\sqrt{25 - x^2}$

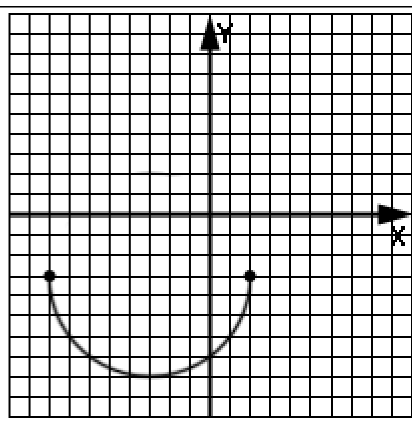
$r^2 = 25$
$r = 5$



D: [-5, 5]
R: [-5, 0]
Reflect across the x-axis

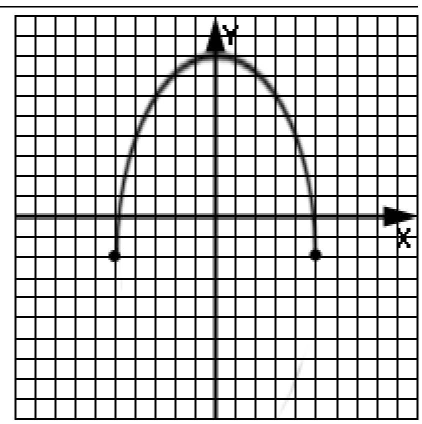
In the following two examples, write the function corresponding to the graph.

Example 9:



$f(x) = -\sqrt{25-(x+3)^2} - 3$

Example 10:



$f(x) = 2\sqrt{25 - x^2} - 2$

Assignment: In problems 1 – 8, graph the transformations, give the domain and range, and give a word description of the transformation from the original parent function.

1. $f(x) = \sqrt{x - 6} - 1$

2. $f(x) = -\sqrt{x} + 3$

3. $f(x) = -3\sqrt{x + 2} + 4$

4. $f(x) = \sqrt{-x + 3} + 4$

5. $f(x) = \sqrt{36 - x^2} + 2$

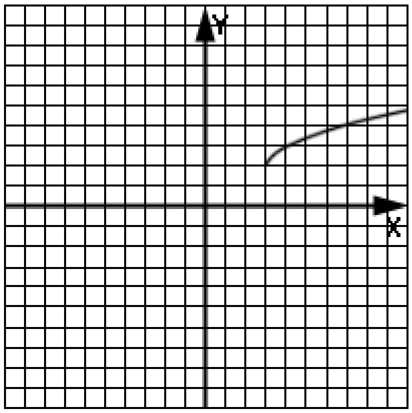
6. $f(x) = -\sqrt{49 - x^2} - 3$

7. $f(x) = 3\sqrt{-x^2 + 10}$

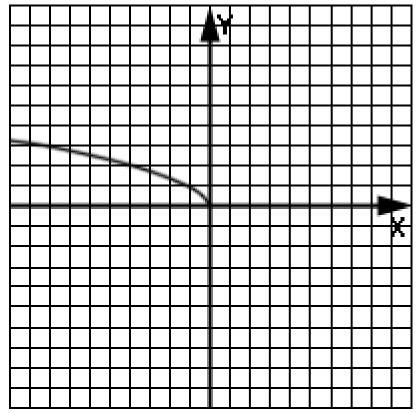
8. $f(x) = -\sqrt{9 - x^2}$

In the following problems write the function corresponding to the graph.

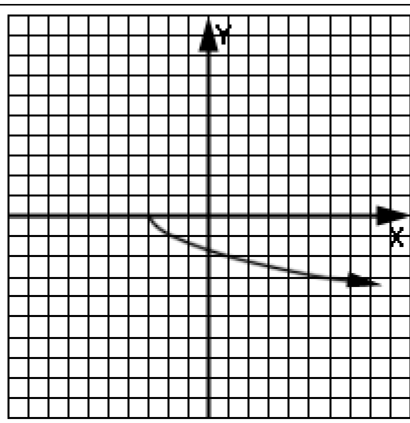
9.



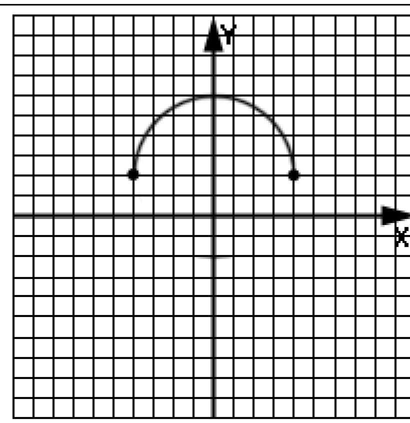
10.



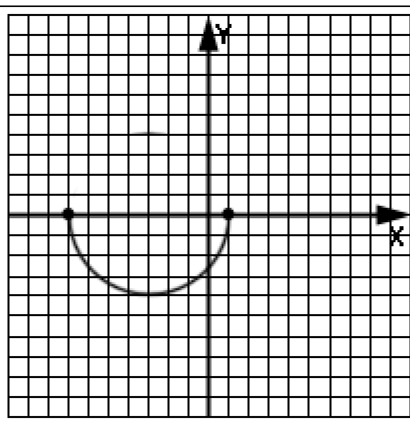
11.



12.



13.



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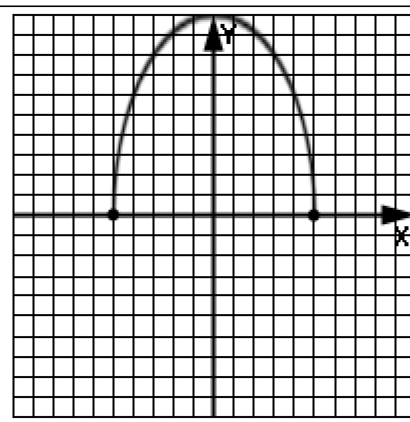
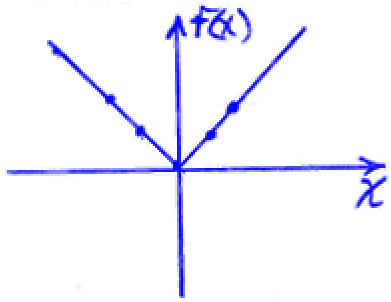


Unit 8: Lesson 02 Absolute value functions (|f(x)| and f(|x|) reflections)

Sketch the **parent function f(x) = |x|** using x = 0, 1, 2, –1, and –2.



x	y
0	0
1	1
2	2
−1	1
−2	2

This basic "V" shape is transformed with reflections and translations just as all the previous functions studied in this course. **Enrichment Topic G** explores the various transformations for this specific parent function.

Here, we will take a different and more general approach to the application of absolute value to functions. This falls under the topic of **"absolute value reflections."**

We will consider when the absolute value is applied to the entire function and, again, when applied to just the argument of the function:

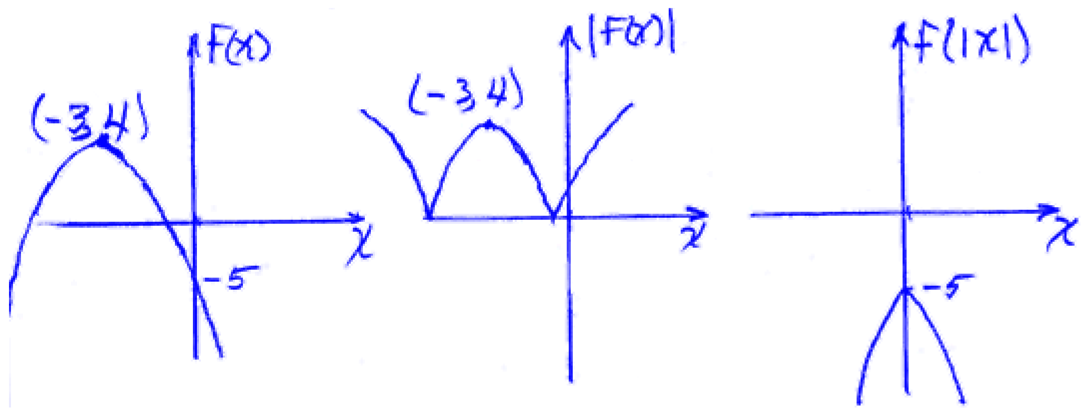
$$|\mathrm{f(x)}| = \begin{cases} f(x) & when\ f(x) \geq 0 \\ -f(x) & when\ f(x) < 0 \end{cases}$$

For function values below the x-axis, erase and replace with their reflection across the x-axis (up).

$$f(|x|) = \begin{cases} f(x) & when\ x \geq 0 \\ f(-x) & when\ x < 0 \end{cases}$$

Erase the graph to the left of the y-axis and replace with the reflection of the right half across the y-axis.

Example 1: Sketch the graph of $f(x) = -(x + 3)^2 + 4$. Sketch $|f(x)|$ and $f(|x|)$.



Verify the three graphs above with a graphing calculator. See **Calculator Appendix V** and an associated video.

Assignment: For each problem, sketch three graphs. Sketch the graph of the given function f(x), |f(x)|, and f(|x|).

1. $f(x) = -3x + 2$

2. $f(x) = 2(x + 4)^2 - 3$

3. $f(x) = \sqrt{x}$

4. $f(x) = \sqrt{x-3} - 2$

5. $f(x) = \sqrt{9-x^2} - 2$

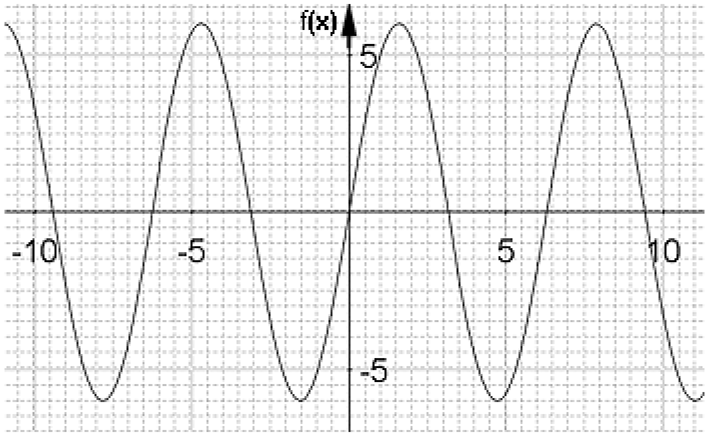
6. $f(x) = -\sqrt{25-x^2}$

7. $f(x) = 4x$

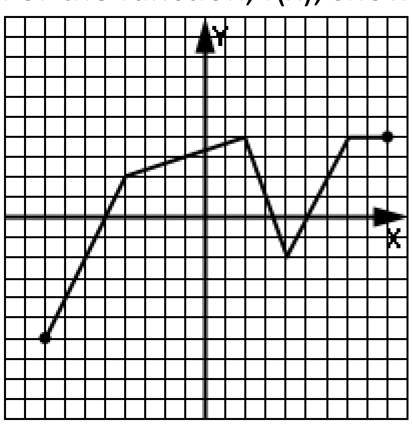
8. $f(x) = -x^2 + 6$

9. $f(x) = \sqrt{-x + 2}$

10. For the function, f(x), shown here, graph |f(x)| and f(|x|).



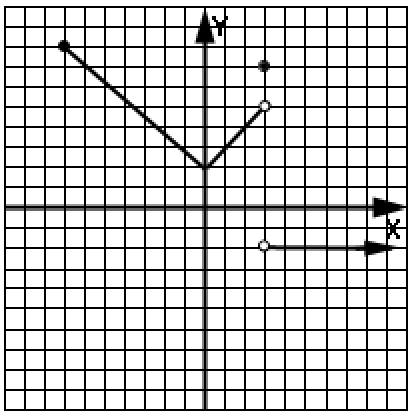
11. For the function, f(x), shown here, graph |f(x)| and f(|x|).



Unit 8: Lesson 03 Piecewise functions, continuity

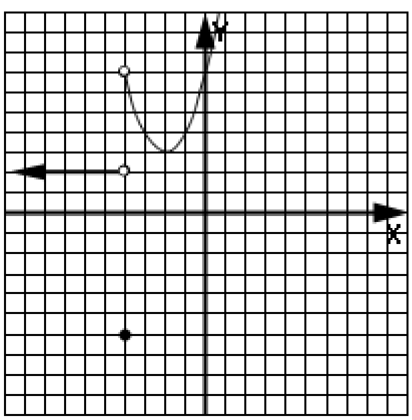
Some functions cannot be expressed as a single **continuous function**. Rather they must be expressed in "pieces." Hence the name **"piecewise functions"**:

Example 1: Write in piecewise form the function graphed below:



$$f(x)=\begin{cases} -\frac{6}{7}x+2 & -7\le x\le 0 \\ x+2 & 0\le x<3 \\ 7 & x=3 \\ -2 & 3<x<\infty \end{cases}$$

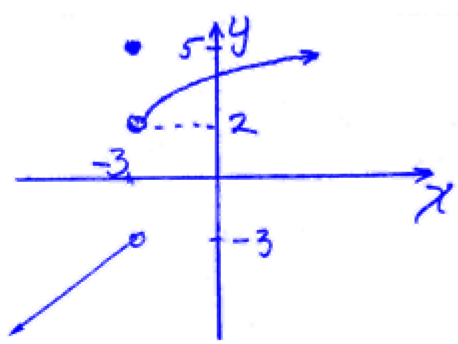
Example 2: Write in piecewise form the function graphed below:



$$f(x)=\begin{cases} 2 & -\infty<x<-4 \\ -6 & x=-4 \\ (x+2)^2+3 & x>-4 \end{cases}$$

Example 3: Graph the following piecewise function:

$$f(x)=\begin{cases} x & when\ x<-3 \\ 5 & when\ x=-3 \\ \sqrt{x+3}+2 & when\ x>-3 \end{cases}$$



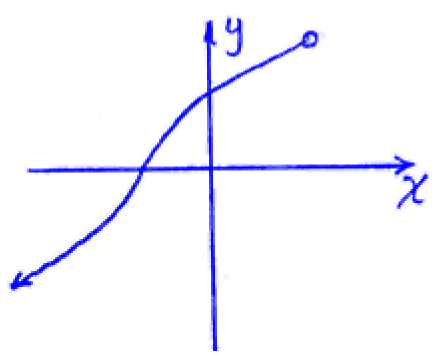
Examples 1, 2, and 3 on the previous page are all examples of **discontinuous functions**. A **continuous function** is one in which, "During the process of graphing the function, it is not necessary to **lift the pencil** in order to continue."

The places of discontinuity are the x-values where it is necessary to "lift the pencil." Later, we will learn more rigorous definitions of continuity.

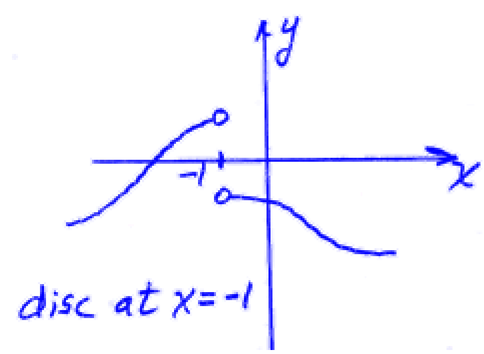
In Examples 4-6, list respectively the places of discontinuity in Examples 1-3.

Example 4:	**Example 5:**	**Example 6:**
$x = 3$	$x = -4$	$x = -3$

Example 7: Draw an example of a continuous function.

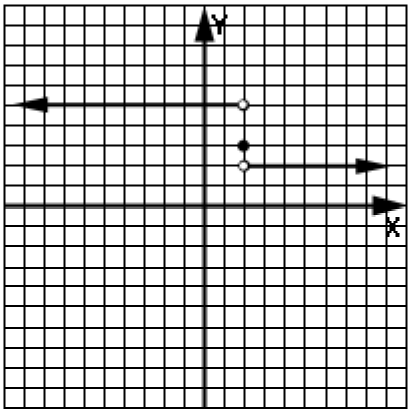


Example 8: Draw an example of a discontinuous function and note the location(s) of any discontinuities.

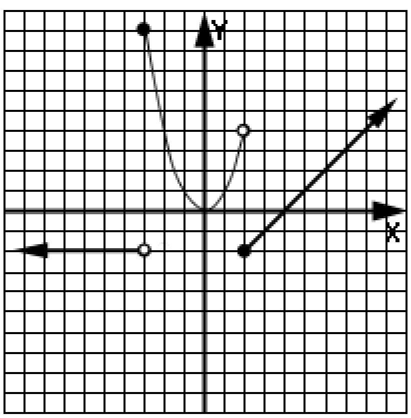


Assignment:

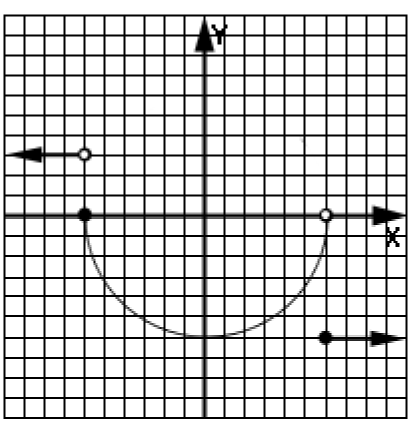
1. Write in piecewise form the function graphed below:



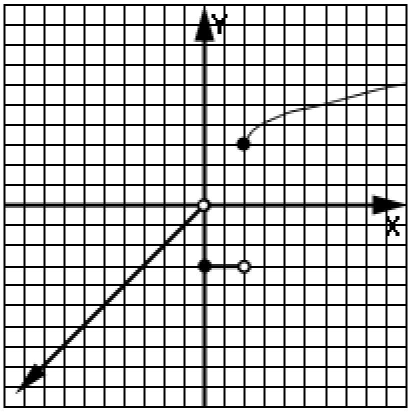
2. Write in piecewise form the function graphed below:



3. Write in piecewise form the function graphed below:



4. Write in piecewise form the function graphed below:



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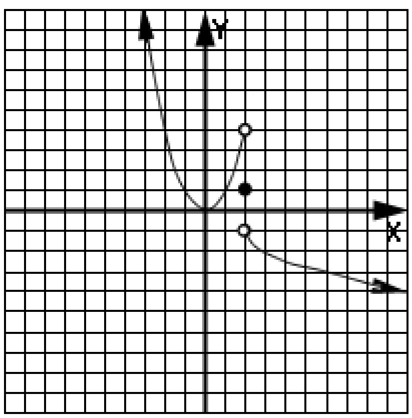
5. Write in piecewise form the function graphed below:



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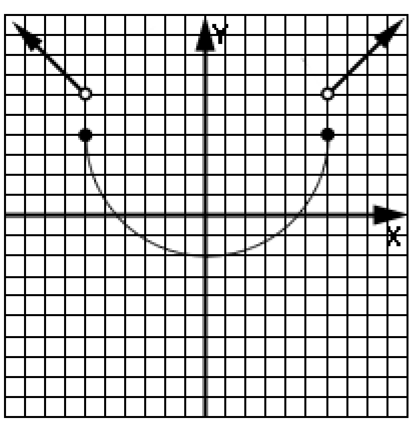
6. Write in piecewise form the function graphed below:



7. Graph the following piecewise function:

$$f(x)\begin{Bmatrix} -5 & \text{when} -\infty < x < 2 \\ 0 & \text{when } x = 2 \\ 7 & \text{when } 2 < x \le 8 \end{Bmatrix}$$

8. Graph the following piecewise function:

$$f(x)\begin{Bmatrix} 5 & \text{when} -\infty < x < -2 \\ x^2 + 3 & \text{when} - 2 \le x \le 2 \\ -\sqrt{x-2} & \text{when } 2 < x < \infty \end{Bmatrix}$$

9. Graph the following piecewise function:

$$f(x)\begin{Bmatrix} -\sqrt{x+2} & \text{when} - 2 \le x < 2 \\ \sqrt{x-2} & \text{when } 2 < x < \infty \\ -1 & \text{when } x = 2 \end{Bmatrix}$$

10. Graph the following piecewise function:

$$f(x)\begin{cases} x & \text{when } -\infty < x < -4 \\ \sqrt{16-x^2} & \text{when } -4 \le x \le 4 \\ -\sqrt{x-4}+2 & \text{when } 4 < x < \infty \end{cases}$$

---

11. Graph the following piecewise function:

$$f(x)\begin{cases} 5 & \text{when } -\infty < x \le -2 \\ \sqrt{16-(x-2)^2} & \text{when } -2 < x \le 6 \\ 5 & \text{when } 6 < x < \infty \end{cases}$$

---

12. Graph the following piecewise function:

$$f(x)\begin{cases} -3 & \text{when } -\infty < x < 2 \\ (x-4)^2+1 & \text{when } 2 \le x \le 6 \\ -3 & \text{when } 6 < x < \infty \end{cases}$$

13. List all the places of discontinuity in problem 9.

14. List all the places of discontinuity in problem 11.

15. Draw an example of a function defined at x = 3 but discontinuous there.

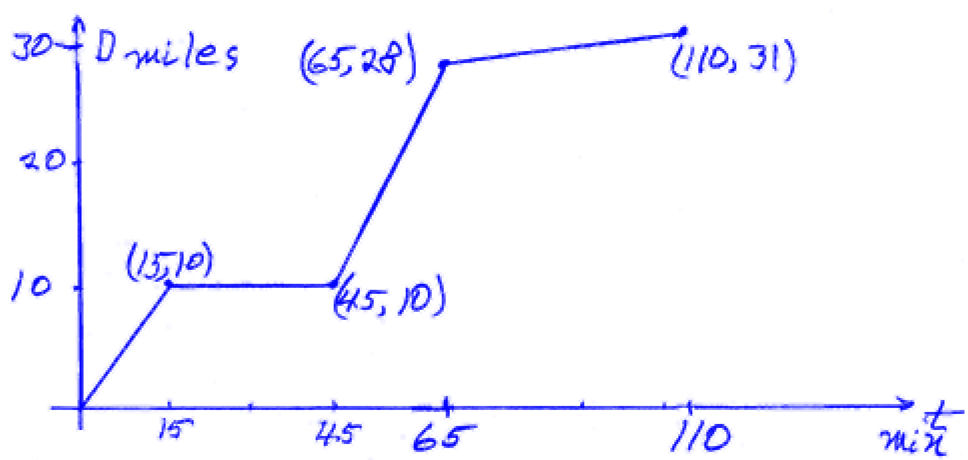
16. Draw an example of a function both undefined and discontinuous at x = 3.

Unit 8: Lesson 04 Rate of change, piecewise word problems

Rate of change generally has to do with **time** and in any event is simply the **slope of a function** at some particular value of the independent variable..

Example 1: Bob starts out for school at 7:00 AM and drives 10 miles in 15 min. He then stops for 30 minutes at a local restaurant for breakfast. Afraid that he might be late for school, he continues driving a little faster for another 20 minutes and covers 18 miles at which point his car breaks down. From that point he jogged the remaining 3 miles to school in 45 min.

Draw a graph with distance on the vertical axis and time on the horizontal axis.



Example 2: What is the rate of change at 12 min?

$$m = \frac{10-0}{15-0}$$

$$= \boxed{\frac{2}{3} \text{ mi/min}}$$

Example 3: What is the rate of change at 17 min?

$\boxed{0}$

slope of a horizontal line is 0.

Example 4: What is the rate of change at 50 min?

$$m = \frac{28-10}{65-45}$$

$$= \boxed{.9 \text{ mi/min}}$$

Example 5: What is the rate of change at 99 min?

$m = \frac{31-28}{110-65}$

$= \boxed{.0\bar{6} \text{ mi/min}}$

Example 6: What is the common meaning of "rate of change" for this problem?

speed

Example 7: What is the rate of change at 65 min?

undetermined

Example 8: Write a piecewise function, D(t), that describes the graph produced in Example 1.

1st segment
$y = mx + b$
$y = \frac{2}{3}x + 0$

2nd segment
$y = 10$

3rd segment
$y = mx + b$
$y = .9x + b$
$10 = .9(45) + b$
$b = -30.5$
$y = .9x - 30.5$

4th segment
$y = mx + b$
$y = .0\bar{6}x + b$ ← sub in (110, 31) $b = 23.666$
$y = .0\bar{6}x + 23.666$

$$D(t) = \begin{cases} \frac{2}{3}t & 0 < t < 15 \\ 10 & 15 < t < 45 \\ .9t - 30.5 & 45 < t < 65 \\ .0\bar{6}t + 23.\bar{6} & 65 < t < 110 \end{cases}$$

Write a transformation for D(t) produced in Example 8 that describes each of the following situations.

Example 9: Bob overslept by 20 min but maintains the same schedule described in Example 1. $D(t-20)$	**Example 10:** Bob starts 20 min earlier but maintains the same schedule described in Example 1. $D(t+20)$
Example 11: It's Bob's birthday and he doesn't care if he is late so he allows himself three times as much time for each part of the journey. $D(\frac{1}{3}t)$	**Example 12:** In a parallel universe another "Bob" lives in a situation in which all distances are twice as far as for our Bob of Example 1. $2\,D(t)$

Assignment:

1. In the beginning a storage tank had 50 liters of liquid in it. One hour later it had 75 liters. A leak went unnoticed for 2 hours and 30 liters were lost. It took 3 hours to repair the leak (The level in the tank remained constant during that time).To make up for lost time because of the leak and repair, another 75 liters were added in the next 2 hours. Make a graph of this scenario with volume as the vertical axis and time as the horizontal axis.

In problems 2-7, use problem 1 to state the rate of change of the volume at the given time.

2. t = .6 hrs	3. t = 2 hrs	4. t = 4.2 hrs

5. $t = 7$ hrs	6. $t = 1$ hr	7. Over what interval is there the fastest rate of change of the volume?

8. Write a piecewise function, $V(t)$, that describes the graph produced in problem 1 above.

Write a transformation for V(t) produced in problem 8 that describes each of the following situations in problems 9-12.

9. Everything takes half as much time.	10. The entire process starts an hour and a half later.
11. All volume measurements are four times as much.	12. The entire process starts an hour earlier.

13. What is the volume of the liquid at 7 hrs and 20 minutes?

Unit 8: Lesson 05 Greatest integer function, power functions

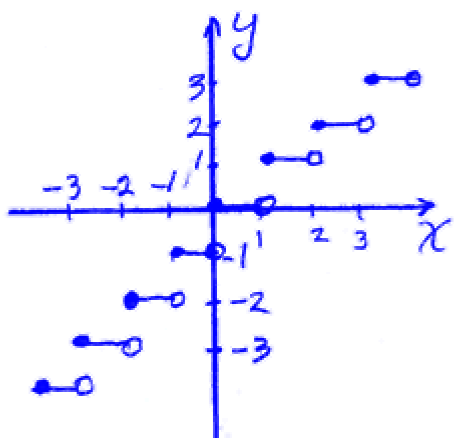
The **greatest integer function** (also called a **step function**) is given by **f(x) =** $\|x\|$ and is defined as the greatest integer less than or equal to *x*.

On the graphing calculator use **Math | Num | 5: int(** . See **Calculator Appendix Z** and an associated video.

Example 1: Find $\|18.6\|$ 18

Example 2: Find $\|-18.6\|$ −19

Example 3: Graph f(x) = $\|x\|$



Example 4: Find the following for f(x) = $\|x\|$:

Domain: All real x

Range: integers

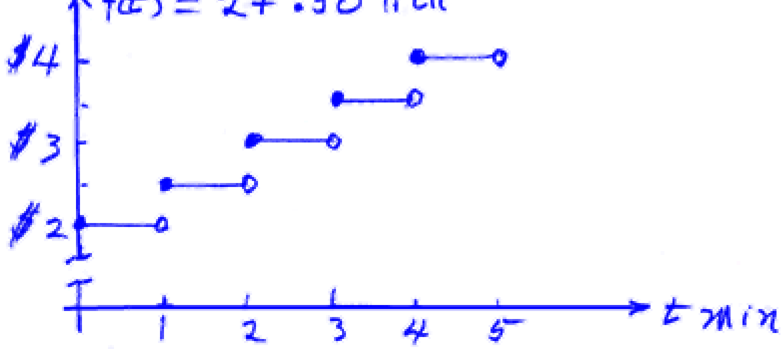
x-intercept: All x values in interval [0, 1)

y-intercept: 0

Discontinuities: at x = all integers

Example 5: A cruise ship charges guests $2.00 for any connect time less than the first full minute in their internet café. For each additional portion of a minute that is just less than a minute, an additional $.50 is charged. Give the function that specifies these charges for any connect time, *t*, in minutes and graph the function.

$f(t) = 2 + .50\,\|t\|$

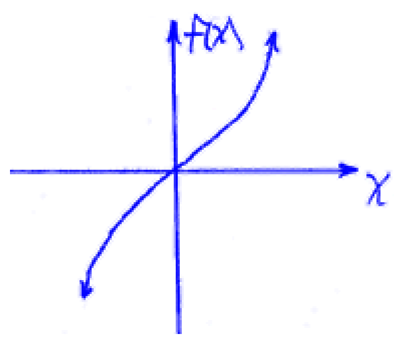


Power functions are of the form **$f(x) = x^n$** where *n* is an integer ≥ 0.

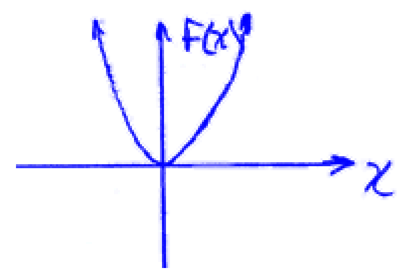
Even values of *n* produce **even functions** all with basically the **same shape**. The larger *n* is, the steeper the curve is except in the interval $-1 < x < 1$ where it gets flatter.

Odd values of *n* produce **odd functions** all with basically the **same shape**. The larger *n* is, the steeper the curve is except in the interval $-1 < x < 1$ where it gets flatter.

Example 6: Sketch the graph of $f(x) = x^3$

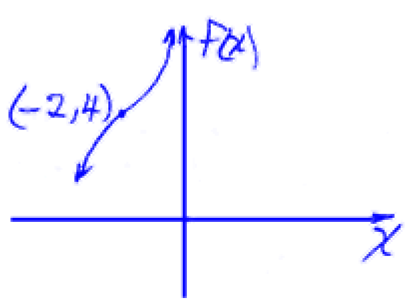


Example 7: Sketch the graph of $f(x) = x^6$



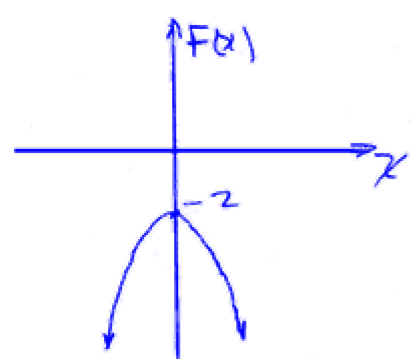
All transformations we have previously studied can be applied to power functions. Sketch the following functions and describe in words the transformation from the original parent function.

Example 8: $f(x) = (x + 2)^5 + 4$



shift left 2, up 4

Example 9: $f(x) = -x^4 - 2$



reflect across x-axis, shift down 2

Assignment:

1. Find the value of $\|4.1\|$.	2. Find the value of $\|-6.7\|$.
3. Find the value of $\|6.2 - 2\|$.	4. Find the value of $3.1 + \|4.1\|$.

5. A speed trap in Nerd City charges a fine of $100 for speeds of 70 up to just less than 75. For each additional increment of 5 mph the fine is an additional $25. Graph the step function *f* on the vertical-axis as the amount of the fine and the speed, *s*, on the horizontal-axis. Restrict the domain to **s ≥ 70**.

Give the function that specifies *f* as a function of *s*. (Hint: Consider using (s – 70)/5 as part of the function.)

6. From problem 5, what is the fine for a speed of 78 mph in an old Ford Pinto?

7. From problem 5, what is the fine for a speed of 187 mph in a Porsche?

8. On the same coordinate system sketch $y = x^3$ and $y = x^7$.

9. On the same coordinate system sketch $y = x^4$ and $y = x^{10}$.

10. Sketch $f(x) = -(x + 2)^3 + 3$

11. Sketch $f(x) = (x - 6)^6 - 2$

12. Give a word description of the transformation in problem 10.

13. Give a word description of the transformation in problem 11.

14. On the same coordinate system sketch $y = -x^5$ and $y = -3x^5$.

15. Sketch $f(x) = (-x + 2)^3$

Unit 8: Cumulative Review

1. Use a graphing calculator to perform a linear regression on the provided data and show the equation of the best-fit line along with a sketch of the line and scatter-plot. See **Calculator Appendices M and N** for a review.

x	y
–70	–40
–29	–52
28	–32
60	–30
89	–25

2. Solve $x^2 - 2x + 3 = 0$ with the quadratic formula.

3. Convert $5^{o}\ 37'\ 51''$ to decimal form.

4. A rectangular field that is adjacent to a river is to be fenced using no more than 500m of fence material. If only three sides are to be fenced (the river serves as one side) what is the maximum area that can be fenced and what are its dimensions?

5. Draw a central angle of a circle whose radius is 14. The length of the arc intercepted by the angle is 20.5. What is the measure of the central angle in radians?

6. Solve the system $5x - 2y = 1$ and $4x - 3y = 4$ with the elimination method.

7. What are the three forms of a quadratic function? Name them as well as giving the functions.

---

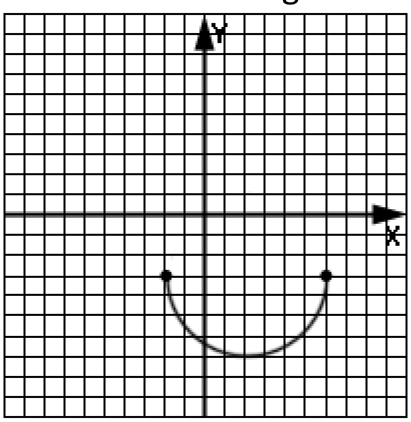
8. Solve $\cos^2\theta - 2\sin\theta = -1$

Unit 8: Review

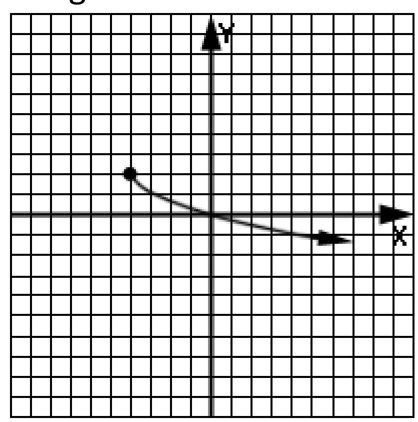
1. Sketch $f(x) = \sqrt{x + 7} - 2$ and describe the transformation.

2. Sketch $f(x) = \sqrt{16 - x^2} - 4$ and describe the transformation.

3. Write the function shown here and give its domain and range.



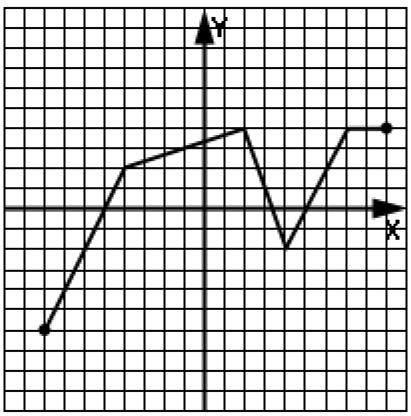
4. Write the function shown here and give its domain and range.



5. Given $f(x) = -(x - 2)^2 + 4$, sketch $|f(x)|$.

6. Given $f(x) = -(x-2)^2 + 4$, sketch $f(|x|)$.

7. Using the given function, sketch $|f(x)|$ and $f(|x|)$.



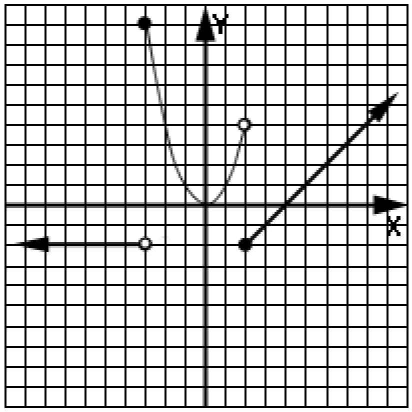
8. Graph the following piecewise function:

$$f(x) = \begin{cases} 5 & \text{when } -\infty < x \le -2 \\ \sqrt{16-(x-2)^2} & \text{when } -2 < x \le 6 \\ 5 & \text{when } 6 < x < \infty \end{cases}$$

9. In problem 8, what are the locations of any discontinuities?

10. Sketch an example of a function that has discontinuities at $x = -2$ and $x = 5$.

11. Write the piecewise function that corresponds to this graph.



12. A policeman walking a beat leaves headquarters and walks 8 blocks in 15 min until he comes to a Crispy Crème Donut Shop where he stays for 30 min. He then continues on his way a distance of 4 blocks in 10 min and realizes he left his wallet at the donut shop. Walking briskly, he returns to the shop in 5 min, and while retrieving his wallet, a call comes in concerning a robbery 8 blocks further down the street. He arrives at the scene of the crime in only 6 minutes. (Use this information for problems 12-17.)

Draw a graph of his position P (in blocks) versus time t (in minutes).

13. Over what interval of time is his speed 0?

14. Over what interval of time is his speed negative?

15. What is the rate of change of his position at 62 minutes?

16. What is the rate of change at 55 minutes?

17. Write a piecewise function that describes the policeman's travels.

18. Evaluate $||-79.6||$

19. Sketch $f(x) = ||x||$

20. Simultaneously sketch $y = x^5$ and $y = x^7$.

21. Simultaneously sketch $y = x^4$ and $y = x^8$.

Pre Calculus, Unit 9

Polynomial Functions

Unit 9: Lesson 01 Polynomial fundamentals, roots, end behavior

A polynomial function (sum of power functions) is of the form:

$$f(x) = ax^n + b\,x^{n-1} + cx^{n-2} + \ldots + dx + e$$

Take note of the following facts:

- All exponents are **positive integers**.
- a, b, c, d, … e are constants.
- The highest power, *n*, is the **degree** of the polynomial.

In the following examples, decide if the function is a polynomial, and if so, what is its degree?

Example 1: $f(x) = 4x(x-1)^2$ $f(x) = 4x(x^2-2x+1)$ $= 4x^3 - 8x^2 + 4x$ yes, degree 3	**Example 2:** $f(x) = 4x^5 + 3x^2 - 2$ yes, degree 5
Example 3: $f(x) = x/6 - 7$ $f(x) = (\frac{1}{6})x - 7$ yes, degree 1	**Example 4:** $f(x) = 5\sqrt[3]{x} - 19$ $f(x) = 5x^{1/3} - 19$ No, fractional exponent
Example 5: $f(x) = 17/x - 9$ $f(x) = 17x^{-1} - 9$ No, neg exponent	**Example 6:** $f(x) = (8x^3 - 3x + 2)/(x - 4)$ No, rational expression

Example 7: $f(x) = 4^x + 11$

No
exponential function

Example 8: $f(x) = 12x^{-1} + 8x + x^2$

No
Neg exponent

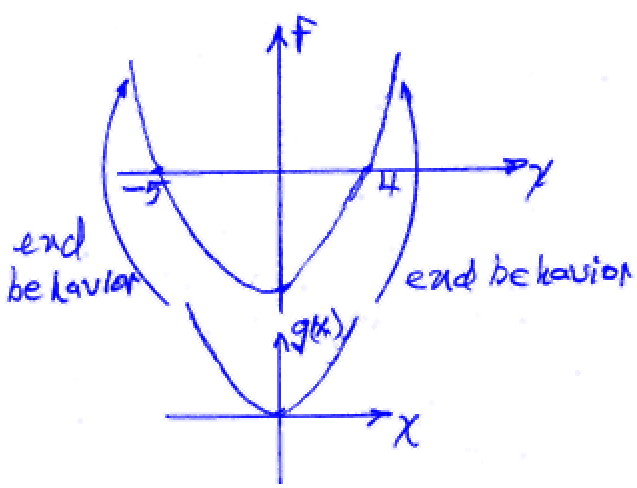
Some additional polynomial facts:

- The graphs of polynomials are **smooth** (no sharp corners) and **continuous** (no gaps or holes).
- The domain is $(-\infty, +\infty)$.
- The **end behavior** (graph to the far left and right) is the same as for a power function of the same degree as the polynomial.
- For a polynomial of degree *n*, there are **at most *n*** **<u>unique</u> roots**.
    - o Single root… graph crosses the x-axis (*f* changes signs)
    - o Double root… graph is tangent to the x-axis (*f* does not change signs)
    - o Triple root… graph crosses x-axis (*f* changes signs). The root is a point of inflection (concavity changes).

Example 9 (single root): Graph $f(x) = (x – 4)(x + 5)$

roots:
$x-4=0$
$x=4$ single roots
$x+5=0$
$x=-5$

$f(x) = (x-4)(x+5)$
$= x^2+x-20$
power function
$g(x) = x^2$



Example 10 (double root): Graph $f(x) = 2x(x - 3)^2$

roots:

$2x(x-3)^2=0$

$2x=0$ single root

$x=0$

double root

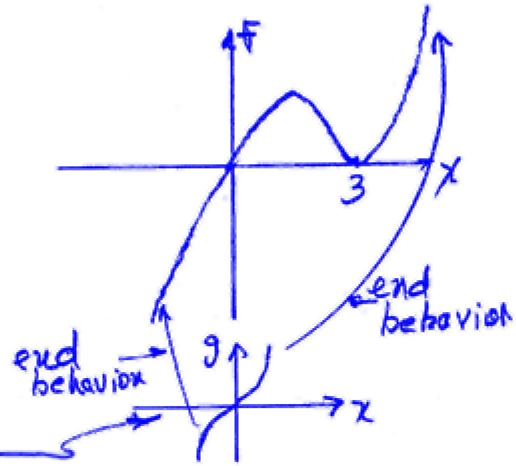
$x-3=0$

$x=3, 3$

$f(x)=2x(x^2-6x+9)$

$=2x^3-12x^2+18x$

Power function $g=2x^3$



Example 11 (triple root): Graph $f(x) = -x(x - 5)^3$

roots:

$-x(x-5)^3=0$

$-x=0$

$x=0$ single root

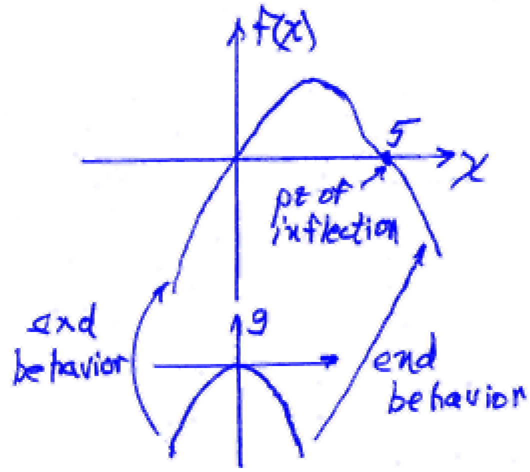
$(x-5)^3=0$ triple root

$x-5=0;\ x=5,5,5$

$f(x)=-x(x^3+\cdots)$

$f(x)=-x^4+\cdots$

Power function $g(x)=-x^4$



Assignment: In the problems 1- 8, decide if the function is a polynomial, and if so, what is its degree?

1. $f(x) = 7x^2 - 3x^{-2}$

2. $f(x) = 4(x)^3 + 1$

3. $f(x) = (7x^3 - 6x + 1)/(x^2 + 2)$

4. $f(x) = 2 - 4/x$

5. $f(x) = \sqrt{x+2} + 7$

6. $f(x) = x/11 - x^2 + 1$

7. $f(x) = x^5 - 13x^2 - x + 1$

8. $f(x) = -2x(x - 2)^2$

Use the roots and end behavior to sketch the following functions. Label the roots.

9. $f(x) = x^3 - 3x^2 - 36x + 108$

10. $f(x) = x^2 - 6x + 9$

11. $f(x) = -x^4 + 16x^2$

12. $f(x) = -x(x^2 - 4)^2$

13. $f(x) = 4x(x - 2)^3$

14. $f(x) = 2x - 2x^2$

15. $f(x) = x^3 - 5x^2$

---

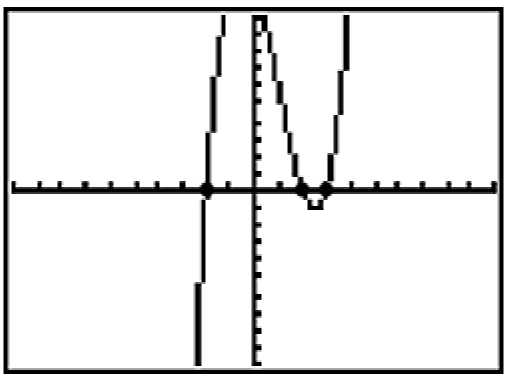
16. $f(x) = x(x^2 - 9)(x^2 - 4)$

Unit 9: Lesson 02 Creating polynomial functions

Given the roots and any other point on the graph of a polynomial, it is possible to write the polynomial function:

In the following examples, use the given information to write the polynomial function. Make a sketch of the function if not already provided.

Example 1: A cubic function contains the point (–3, –30) and produces the graph shown below. Confirm your solution with a graphing calculator.



Single roots at $-2, 2, 3$

$y = a(x+2)(x-2)(x-3)$

sub in $(-3,-30)$

$-30 = a(-3+2)(-3-2)(-3-3)$

$-30 = a(-1)(-5)(-6)$

$-30 = a[-30]$

$1 = a$

$\boxed{y = 1(x+2)(x-2)(x-3)}$

Example 2: A cubic function contains the point (–4, 3) and has a double root at –3 and a single root at 2.

$y = a(x+3)^2(x-2)$

sub in $(-4, 3)$

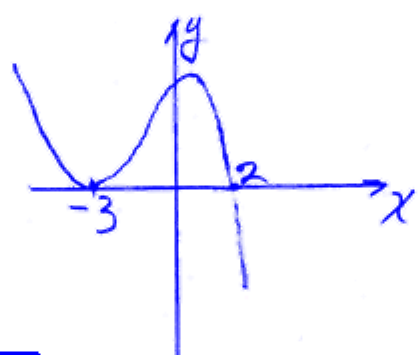
$3 = a(-4+3)^2(-4-2)$

$3 = a(-1)^2(-6)$

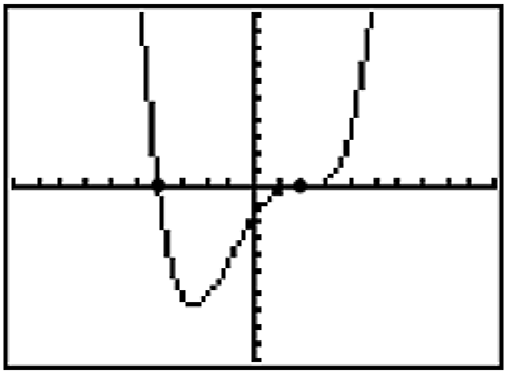
$3 = -6a$

$-\frac{1}{2} = a$

$\boxed{y = -\frac{1}{2}(x+3)^2(x-2)}$



Example 3: A fourth degree function contains the point (–6, 51) and has roots as shown below. Confirm your solution with a graphing calculator.



Single root at -4
Triple root at 2

$y = a(x+4)(x-2)^3$

$\boxed{y = .0498(x+4)(x-2)^3}$

sub in $(-6, 51)$

$51 = a(-6+4)(-6-2)^3$

$51 = a(-2)(-8)^3$

$51 = a(-2)(-512)$

$a = .0498$

Assignment: In the following problems, use the given information to write the polynomial function. Make a sketch of the function if not already provided.

1. A cubic function contains the point (0, 100) and has a double root at –7, and a single root at –1.

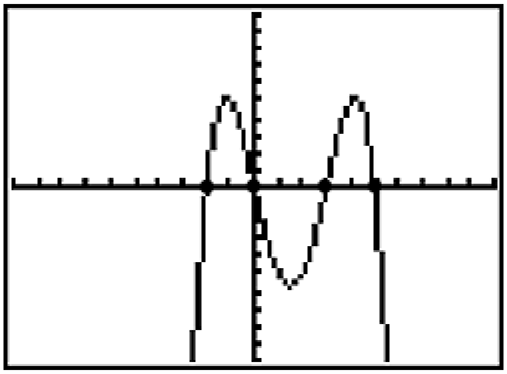
2. A cubic function contains the point (3, 32) and has roots at –3, –2, and 2.

3. A cubic function contains the point (2, 8) and has roots at 0, 1, and 3.

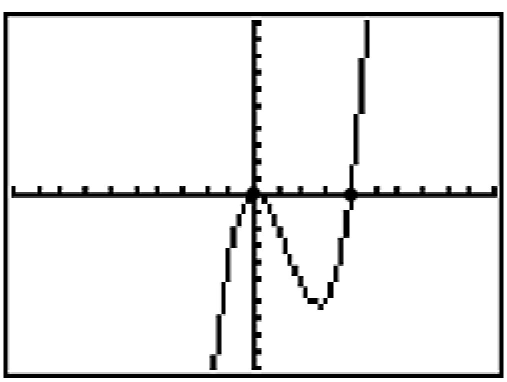
4. A cubic contains the point (2, 4), and has a double root at –2 and a single root at 3.

5. A quartic (4^{th} degree equation) contains the point (–1, 4) and has a triple root at –2 and a single root at 3.

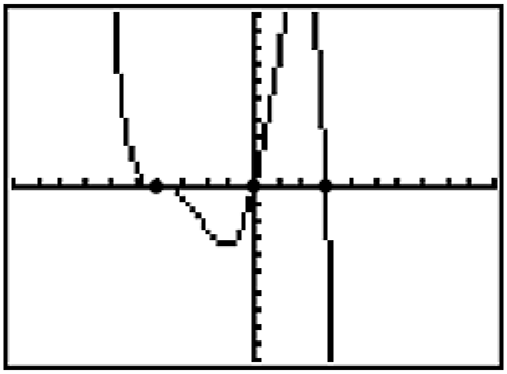
6. The function contains the point (1, –5) and produces the graph shown below. Confirm your solution with a graphing calculator.



7. The function contains the point (1, –2) and produces the graph shown below. Confirm your solution with a graphing calculator.



8. The function contains the point (1, 7.5) and produces the graph shown below. Confirm your solution with a graphing calculator.



Unit 9: Lesson 03 Long division (factoring, finding roots)

Consider the simple task of dividing 12 by 4. Since there is **no remainder**, this means 4 divides evenly into 12 and is, therefore, a **factor** of 12.

Similarly, with polynomials it can be shown with **long division** that $(x + 2)$ is a factor of $f(x) = 2x^3 + 3x^2 - 8x - 12$. Since $x + 2$ is a factor, it follows that $x = -2$ is a root.

Example 1: Divide $(x + 2)$ into $f(x) = 2x^3 + 3x^2 - 8x - 12$ to demonstrate that it is a factor and that $x = -2$ is a root.

$$2x^2 - x - 6$$
$$x+2 \,\overline{)\, 2x^3 + 3x^2 - 8x - 12}$$
$$\ominus 2x^3 \ominus 4x^2$$
$$-x^2 - 8x$$
$$\oplus x^2 \oplus 2x$$
$$-6x - 12$$
$$\oplus 6x \oplus 12$$
$$0$$

$\leftarrow$ remainder is 0, so $x+2$ is a factor

Find roots

$$f(x) = (2x^2 - x - 6)(x+2) = 0$$
$$2x^2 - x - 6 = 0 \qquad x + 2 = 0$$
$$x = \boxed{-2}$$

Example 2: Completely factor f(x) from Example 1 and find the other two roots.

$$f(x) = (2x^2 - x - 6)(x+2) = 0$$
$$2x^2 - x - 6 = 0$$
$$(2x+3)(x-2) = 0$$
$$f(x) = \boxed{(2x+3)(x-2)(x+2)}$$

$$2x + 3 = 0 \qquad x - 2 = 0$$
$$2x = -3 \qquad x = \boxed{2}$$
$$x = \boxed{-3/2}$$

Assignment: In problems 1- 4, use long division to determine if f(x) is divisible by the g(x). If factorable, completely factor f(x) and find all of the roots of f(x).

1. $f(x) = 6x^3 - x^2 - 5x + 2$; $g(x) = 3x - 2$

---

2. $f(x) = -80x^2 + 48x^3 + 41x - 6$; $g(x) = -2 + 3x$

3. $f(x) = x^5 - 3x^3 + 8x^2 - 24$; $g(x) = x^2 - 3$

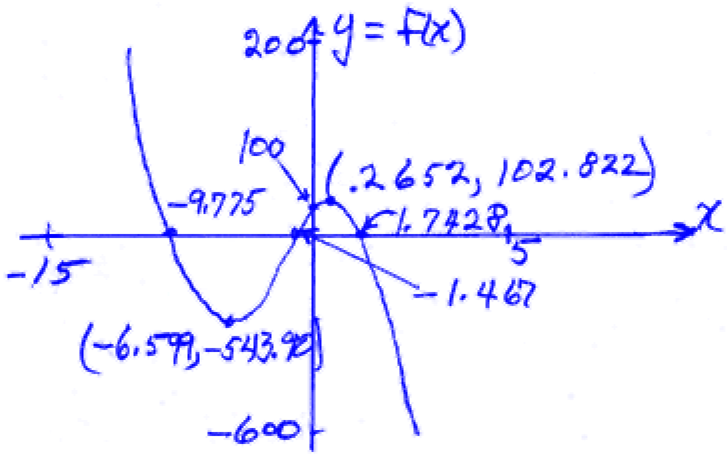
4. $f(x) = 4x^3 + x^2 - 8x + 10$; $g(x) = 4x + 5$

5. Determine k so that $x - 4$ is a factor of $x^2 - 6x + k$.

Unit 9: Lesson 04 **Analyzing polynomials with a graphing calculator**

In the following example, use a graphing calculator to sketch the polynomial. Label local maxima points, local minima points, y intercept, and roots. (Note that the x and y axes can be scaled differently.)

Example 1: $f(x) = -4x^3 - 38x^2 + 21x + 100$



Use f(x) from Example 1 in the following examples:

Example 2: Evaluate f(3.2).

-352.992

Example 3: Give the intervals over which $f(x) < 0$.

$(-9.775, -1.467), (1.7428, \infty)$

Example 4: Where is $f(x) = 250$?

$Y1 = f(x)$
$Y2 = 250$ Find intersection
$x = \boxed{-10.35655}$

Example 5: Where is $f(x) = 0$?

roots
$x = \boxed{-9.775, -1.467, 1.7428}$

This lesson involves finding roots, intersection points, minima, and maxima. If review is needed on how to find these things with a graphing calculator, consult the following Calculator Appendices and associated videos:

Intersection points: **Calculator Appendix C**
Roots: **Calculator Appendix I**
Maxima and Minima: **Calculator Appendix J**

Assignment: In problems 1, 6, and 11, use a graphing calculator to sketch the polynomial. Label local maxima points, local minima points, y intercept, and roots.

1. $f(x) = -x^4 + 3x^2 - 3x + 6$

Use f(x) from problem 1 in problems 2-5:

2. Evaluate f(2.71)	3. Where is f(x) = 8?
4. Give the intervals over which $f(x) \geq 0$.	5. Give the interval(s) over which f(x) is increasing.

6. $f(x) = x^4 + x^3 - 5x^2 + 2$

Use f(x) from problem 6 in problems 7-10:

7. Evaluate f(5)	8. Where is $f(x) = -20$?
9. Give the interval(s) over which f(x) is decreasing.	10. Give the intervals over which $f(x) > 3$.

11. $f(x) = x^4 - 3x^3 - 2x + 3$

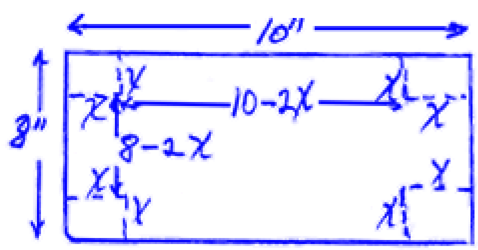
Use f(x) from problem 6 in problems 12-15:

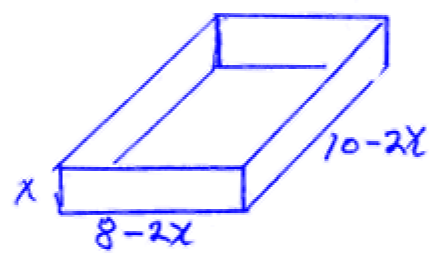
12. What is the value of y when $x = -1.5$?	13. Where does there appear to be point(s) of inflection?
14. Where is $f(x) = 7$?	15. What is (are) the interval(s) over which $f(x) \leq 0$?

Unit 9: Lesson 05 Applications (maximizing volume)

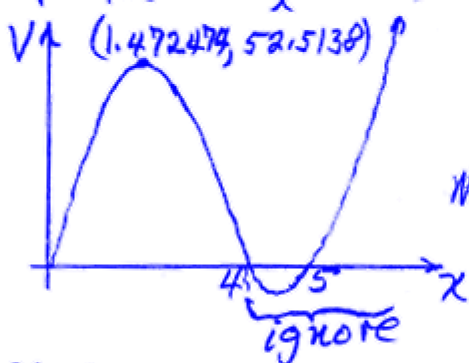
Example 1:(box without a lid): A box without a lid is to be made from a rectangular sheet of card board (8” X 10”) by cutting out the same small square from each of the four corners and then folding up the sides to form a three-dimensional rectangular box.

- Make a diagram of the flat sheet with the squares cutout of each corner.
- Make a diagram of the 3-D box
- Write a volume function in terms of *x*, the dimension of the square.
- Use a graphing calculator to make a sketch of the volume versus *x*.
- Determine the size of the squares that are cut away so as to maximize the volume of the box.
- What is the resulting maximum volume of the box?





$V = x(8-2x)(10-2x)$



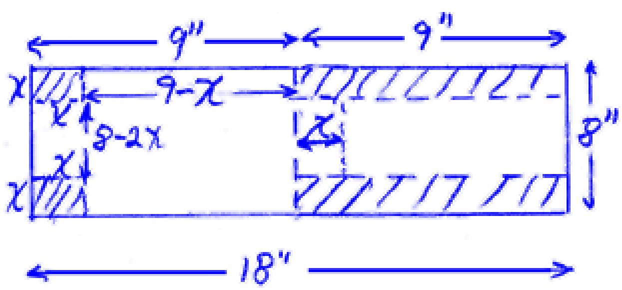
$x = \boxed{1.472474 \text{ in}}$

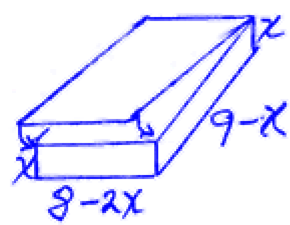
Max Volume $= \boxed{52.5138 \text{ in}^3}$

Xmin = 0
Xmax = 15
Ymin = -10
Ymax = 60

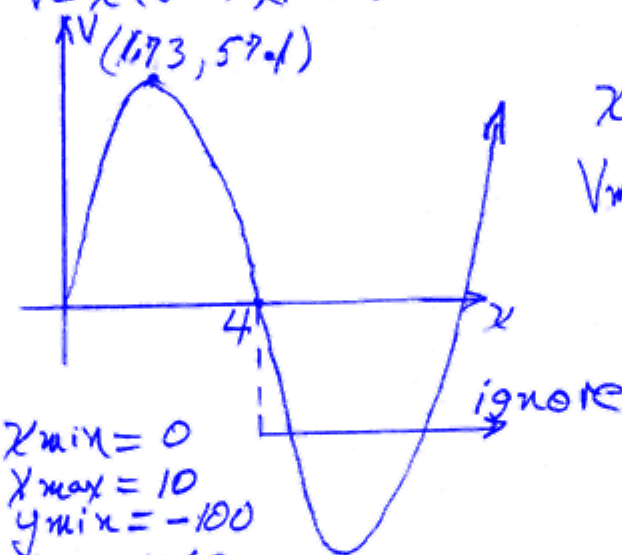
Example 2: (box with a lid): A box with a lid is to be made from a rectangular sheet of card board (18” X 8”) by cutting out the same small square from the two corners on one of the 8” sides and long rectangular cutouts from the opposite corners. Fold up the sides to form a three-dimensional rectangular box.

- Make a diagram of the flat sheet with the cutouts.
- Make a diagram of the 3-D box
- Write a volume function in terms of *x*, the common dimension of the cutouts.
- Use a graphing calculator to make a sketch of the volume versus *x*.
- Determine *x* so as to maximize the volume of the box.
- What is the resulting maximum volume of the box?





$V = x(8-2x)(9-x)$



$x = \boxed{1.73 \text{ in}}$

$V_{max} = \boxed{57.1 \text{ in}^3}$

xmin = 0
xmax = 10
ymin = -100
ymax = 60

Assignment:

1. A box **without a lid** is to be made from a rectangular sheet of card board (10” X 14”) by cutting out the same small square from each of the four corners and then folding up the sides to form a three-dimensional rectangular box.

- Make a diagram of the flat sheet with the squares cutout of each corner.
- Make a diagram of the 3-D box
- Write a volume function in terms of *x*, the dimension of the square.
- Use a graphing calculator to make a sketch of the volume versus *x*.
- Determine the size of the squares that are cut away so as to maximize the volume of the box.
- What is the resulting maximum volume of the box?

2. A box **with a lid** is to be made from a rectangular sheet of card board (20” X 10”) by cutting out the same small square from the two corners on one of the 10” sides and long rectangular cutouts from the opposite corners. Fold up the sides to form a three-dimensional rectangular box.

- Make a diagram of the flat sheet with the cutouts.
- Make a diagram of the 3-D box
- Write a volume function in terms of *x*, the common dimension of the cutouts.
- Use a graphing calculator to make a sketch of the volume versus *x*.
- Determine *x* so as to maximize the volume of the box.
- What is the resulting maximum volume of the box?

3. Suppose we are to make a rectangular box in which the sum of the perimeter of the base and the height is not to exceed 100.” Furthermore, one side of the base is 4 times the other side of the base. What are the dimensions of the box that would result in a maximum volume? What is the maximum volume?

---

4. Suppose a right-circular cylinder has restrictions imposed on it so that the perimeter of the base plus the height cannot exceed 100.” What are the dimensions that would produce the greatest volume? What is the volume?

Unit 9: Cumulative Review

1. Solve the triangle ABC where A = 66°, B = 109°, a = 12.

2. Simplify $\dfrac{\frac{1}{x} - \frac{5}{2x}}{\frac{1}{6x^2} - 2}$

3. Sketch $f(x) = -\sqrt{25 - x^2} - 2$

4. Sketch $f(x) = \sqrt{4 + x} - 2$

5. $f(x) = (x + 6)^2 - 2$; sketch $|f(x)|$

6. $f(x) = (x + 6)^2 - 2$; sketch $f(|x|)$

7. Find the vertex of the parabola given by $f(x) = 4x^2 - 6x + 1$.

8. Find the vertex of the parabola given by $f(x) = 3(x + 5)(x - 7)$.

9. Using algebraic tests, determine if $f(x) = 4x^3 - 2x^2 + 1$ is an even or an odd function (or perhaps neither).

10. Define all six trig functions in terms of x, y, and r.

11. Define all six trig functions in terms of opp, adj, and hyp.

Unit 9: Review

In problems 1-4, determine if the function is a polynomial. If not, give a reason.

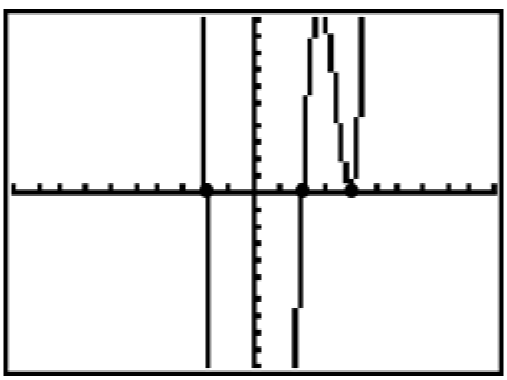
1. $f(x) = 3x(x + 2)^2$	2. $f(x) = 1/(x + 3)$
3. $f(x) = x^3 + x^2 + x^{-1}$	4. $f(x) = \sqrt{x^2 + 7} - 2$

5. Using roots and "end behavior", sketch $f(x) = 2x(x - 3)^2$.

6. Using roots and "end behavior", sketch $f(x) = (x + 1)(x - 6)^3$.

7. Write the polynomial function that contains the point (–4, 1), has a triple root at 3, and has a single root at –2. Sketch the graph.

8. Write the polynomial function that contains the point (0, –128) and produces the graph shown here.



9. Use long division to show that x + 2 is a factor of $f(x) = 2x^3 + 3x^2 - 8x - 12$. Completely factor f(x).

10. Determine *k* so that $x - 5$ is a factor of $x^2 - 15x + k$.

11. Use a graphing calculator to find the local maxima & minima, y-intercept, and roots of $f(x) = -4x^3 - 36x^2 + 22x + 90$. Sketch the graph and label.

12. From problem 11, find $f(-2.07)$.	13. From problem 11, list the interval(s) over which $f(x) \geq 0$.
14. From problem 11, list the interval(s) over which $f(x)$ decreases.	15. From problem 11, find where $f = -100$.

Sem 1: Review

Comprehensive Review

(Use calculators only on highlighted problems.)

1. Give the six trig functions in terms of *x*, *y*, and *r*.

2. Give the six trig functions in terms of opp, adj, and hyp.

3. Convert 3^{o} 16’ 4” to decimal degrees.

4. Convert 187.3782^{o} to degrees, minutes, and seconds.

5. Convert 168^{o} to radians.

6. Convert $3\pi/7$ radians to degrees.

7. If $\cos\theta = 5/13$ and θ is in the 4^{th} quadrant, find $\sin\theta$ and $\tan\theta$.

8. Without using a calculator, give the value of these trig functions: $\sin(180^o) =$ $\cos(90^o) =$ $\csc(270^o) =$ $\cot(0^o) =$	9. Draw a 30^o-60^o-90^o triangle and label the standard lengths of the sides.
10. Draw a 45^o-45^o-90^o triangle and label the standard lengths of the sides.	11. Use a calculator to find $\sec(5\pi/9$ radians).

12. Find the horizontal and vertical components of a force vector having an angle of elevation of 13^o and a magnitude of 202 lb.

13. Solve triangle ABC where $A = 16^o$, $B = 57^o$, $a = 156.9$. Find the area.

---

14. Solve triangle ABC where $c = 100$, $B = 62^o$, and $b = 91$.

---

15. Solve triangle ABC where $a = 5$, $c = 12$, and $B = 28^o$.

16. Two ropes are horizontally pulling on a tree. The first has a magnitude of 320 newtons and the second, 560 newtons. The angle between the forces is 36^{o}. What is the magnitude of the resultant and what angle does it make with the 560 newton force?

---

17. Show that (sec A + 1)(sec A – 1) simplifies to $\tan^2 A$.

---

18. Prove $\frac{\text{cscA}-\text{sinA}}{\text{cotA}} - \frac{\text{cotA}}{\text{cscA}} = 0$

19. Find $\cos(A/2)$ when $\cos A = -4/5$, $\pi/2 \le A \le \pi$.

---

20. Find $\sin(A + B)$ where $\sin A = -2/5$, $\pi \le A \le 3\pi/2$; $\cos B = 1/5$, $3\pi/2 \le B \le 2\pi$.

---

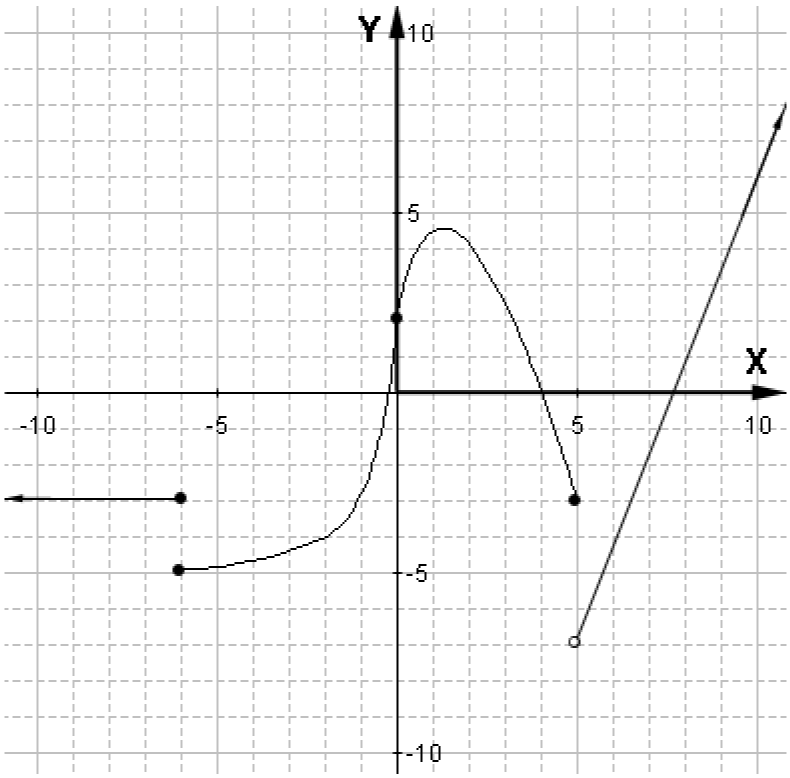
21. Find $\tan(B/2)$ when $\sin B = -2/5$, $3\pi/2 \le B \le 2\pi$.

22. Prove $\frac{\cos 3A - \cos 5A}{\sin 3A + \sin 5A} = \tan A$

23. Solve $\tan^2\theta + \tan\theta = 0$.

24. Solve $2\cos\theta - 5 + 2\sec\theta = 0$.

25.



a. Is this a function?

b. Y-intercept?

c. Zeros?

d. f(5)?

e. One-to-one?

f. f(0)?

g. Increasing interval(s)?

h. Decreasing interval(s)?

i. Constant interval(s)?

j. Intervals where $f(x) \geq 0$

k. Domain?

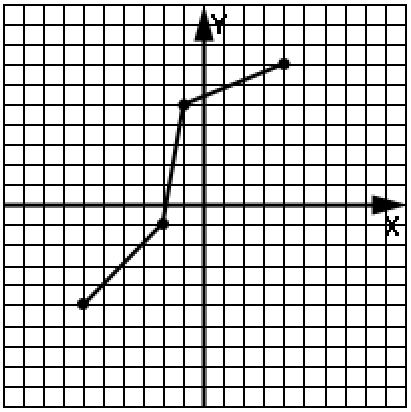
l. Range?

26. $f(x) = 3x^2 - 7x + 1$; Find $f(-1)$

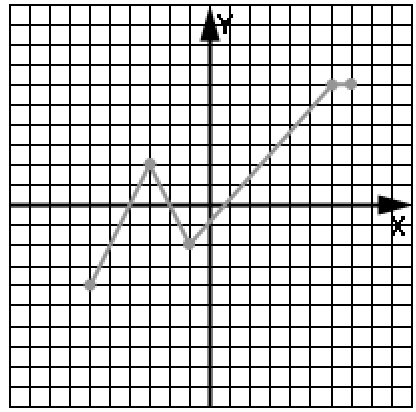
27. Find $(f \circ g)(x)$ where $f(x) = 3x + 1$ and $g(x) = 1/x$.

28. Determine the domain, x-intercept, and y-intercept for $y = (x + 2)/(x^2 - 16)$.

29. Sketch the reflection of the function across the y-axis:



30. Draw the transformed function, f(2x), and describe the transformation in words.

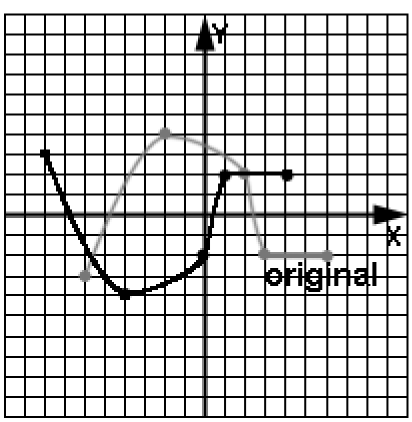


The original function is f(x).

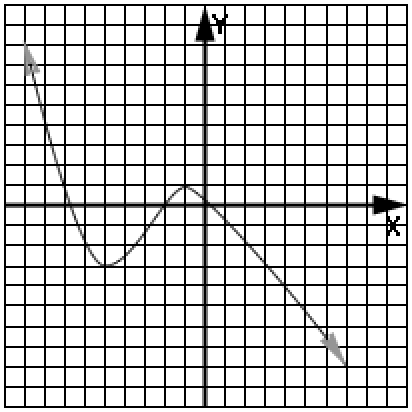
31. Use the table to determine if the function f(x) is even, odd, or neither.

x	f(x)	f(–x)	–f(–x)
1	4	–4	4
2	8	–8	8
3	32	–32	32
–1	–18	18	–18
–2	–20	20	–20

32. Give the functional transformation from the lighter to the darker function and the word description of the transformation.



33.



What are the locations of the local maxima?

Justify the above answer.

What are the local maxima?

What are the locations of the local minima?

Justify the above answer.

What are the local minima?

What is the absolute maximum?

What is the absolute minimum?

Give the interval(s) over which the function is decreasing:

Give the interval(s) over which the function is increasing:

34. Sketch $f(x) = \sqrt{x-6} + 5$

35. Sketch $f(x) = \sqrt{36 - x^2} + 1$

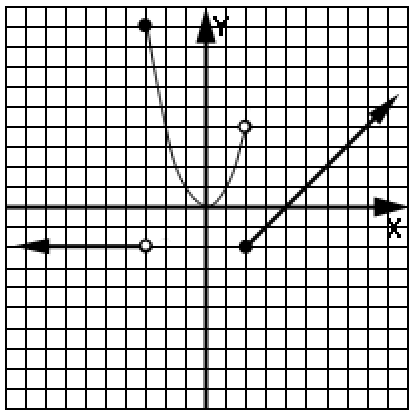
36. For the function, $f(x) = x^2 + 4x – 60$, find the vertex, roots, axis of symmetry, and the y-intercept. Sketch a graph of the function and label completely.

37. A rancher intends to build a corral of maximum area; however, only 320 ft of fence is available. One side of the corral is to have a 10 ft gap where he will eventually put a gate. What will be the maximum area he can enclose and what will be the dimensions of the rectangular corral?

38. In a college dorm Jamie is on the 4th floor (20 m above the ground). Adam is on the 5th floor (25 m above the ground). Simultaneously, Jamie releases a rock and Adam throws his rock downward with a speed of 5 m/sec.

(a) Give the position of each person's rock as a function of time.	(b) On the same coordinate system, graph the vertical positions of both rocks above ground as functions of time.
(c) When does Adam's rock pass the 4th floor?	(d) When are the two rocks at the same height?
(e) When is Jamie's rock 10 m above the ground?	(f) How much later after Adam's rock strikes the ground does Jamie's rock reach the ground?

39. Write in piecewise form the function graphed below:



40. Use the roots and end behavior to sketch $f(x) = -x^4 + 16x^2$. Label the roots.

41. A cubic function contains the point (2, 4), and has a double root at –2 and a single root at 3. Write the function and sketch it.

42. Use long division to determine if $f(x) = 6x^3 – x^2 – 5x +2$ is divisible by the $g(x) = 3x – 2$. If factorable, completely factor f(x) and find all of the roots of f(x).

43. A box **without a lid** is to be made from a rectangular sheet of cardboard (10" X 14") by cutting out the same small square from each of the four corners and then folding up the sides to form a three-dimensional rectangular box. Determine the size of the squares that are cut away so as to maximize the volume of the box. What is the resulting maximum volume of the box?